12 CRITICAL PATH ANALYSIS

Objectives

After studying this chapter you should

- be able to construct activity networks;
- be able to find earliest and latest starting times;
- be able to identify the critical path;
- be able to translate appropriate real problems into a suitable form for the use of critical path analysis.

12.0 Introduction

A complex project must be well planned, especially if a number of people are involved. It is the task of management to undertake the planning and to ensure that the various tasks required in the project are completed in time.

Operational researchers developed a method of scheduling complex projects shortly after the Second World War. It is sometimes called **network analysis**, but is more usually known as **critical path analysis** (**CPA**). Its virtue is that it can be used in a wide variety of projects, and was, for example, employed in such diverse projects as the Apollo moonshot, the development of Concorde, the Polaris missile project and the privatisation of the electricity and water boards. Essentially, CPA can be used for any multi-task complex project to ensure that the complete scheme is completed in the minimum time.

Although its real potential is for helping to schedule complex projects, we will illustrate the use of CPA by applying it to rather simpler problems. You will often be able to solve these problems without using CPA, but it is an understanding of the concepts involved in CPA which is being developed here.

12.1 Activity networks

In order to be able to use CPA, you first need to be able to form what is called an **activity network**. This is essentially a way of illustrating the given project data concerning the tasks to be completed, how long each task takes and the constraints on the order in which the tasks are to be completed. As an example, consider the activities shown below for the construction of a garage.

	activity	duration (in days)
A	prepare foundations	7
B	make and position door frame	2
C	lay drains, floor base and screed	15
D	install services and fittings	8
E	erect walls	10
F	plaster ceiling	2
G	erect roof	5
H	install door and windows	8
I	fit gutters and pipes	2
J	paint outside	3

Clearly, some of these activities cannot be started until other activities have been completed. For example

activity G - erect roof

cannot begin until

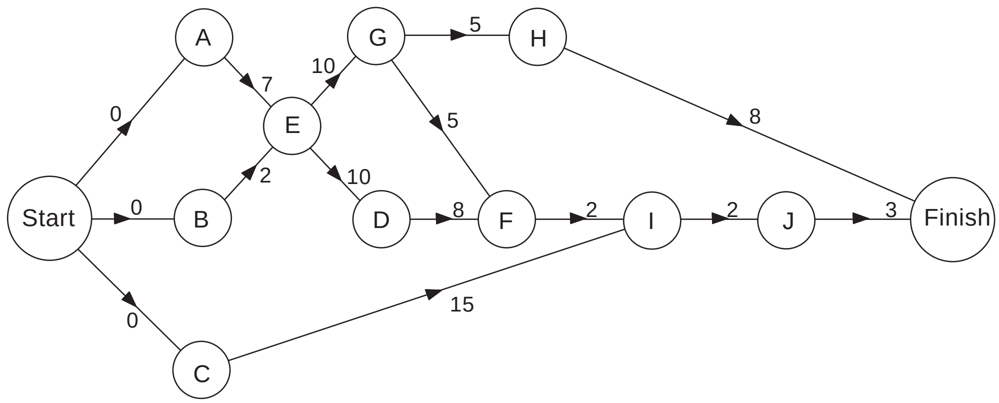
activity E - erect walls

has been completed. The following table shows which activities must precede which.

D must follow E
E must follow A and B
F must follow D and G
G must follow E
H must follow G
I must follow C and F
J must follow I.

We call these the **precedence relation**s.

All this information can be represented by the network shown below.



In this network

each **activity** is represented by a **vertex**;

joining vertex X to vertex Y shows that activity X must be **completed** before Y can be started;

the **number** marked on each arc shows the **duration** of the activity from which the arc starts.

Note the use of 'arc' here to mean a directed edge. Sometimes we can easily form the activity network, but not always, so we need to have a formal method. First try the following activity.

Activity 1 Making a settee

A furniture maker is going to produce a new wooden framed settee with cloth-covered foam cushions. These are the tasks that have to be done by the furniture maker and his assistants and the times they will take :

	activity	time in days
A	make wooden arms and legs	3
B	make wooden back	1
C	make wooden base	2
D	cut foam for back and base	1
E	make covers	3
F	fit covers	1
G	put everything together	1

Each activity can only be undertaken by one individual.

The following list gives the order in which the jobs must be done:

B	must be after	C
A	must be after	B and C
D	must be after	B and C
E	must be after	D
F	must be after	E
G	must be after	A, B, C, D, E and F

Construct an appropriate activity network to illustrate this information.

12.2 Algorithm for constructing activity networks

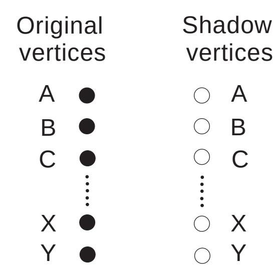
For simple problems it is often relatively easy to construct activity networks but, as the complete project becomes more complex, the need for a formal method of constructing activity networks increases. Such an algorithm is summarised below.

Start Write down the original vertices and then a second copy of them alongside, as illustrated on the right. If activity Y must follow activity X draw an arc from original vertex Y to shadow vertex X. (In this way you construct a **bipartite graph**.)

Step 1 Make a list of all the original vertices which have **no** arcs incident to them.

Step 2 Delete all the vertices found in Step 1 and their corresponding shadow vertices and all arcs incident to these vertices.

Step 3 Repeat Steps 1 and 2 until all the vertices have been used.



The use of this algorithm will be illustrated using the first case study, constructing a garage, from Section 12.1.

The precedence relations are:

D must follow E

E must follow A and B

F must follow D and G

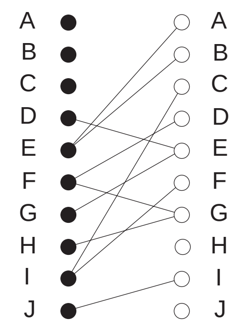
G must follow E

H must follow G

I must follow C and F

J must follow I

These are illustrated opposite.



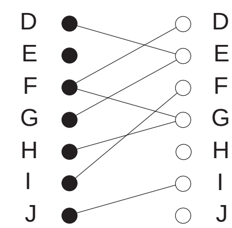
Applying the algorithm until all vertices have been chosen is shown below.

Step 1 - original vertices with no arcs

A, B, C

Step 2 - delete all arcs incident on A, B, C and redraw as shown

Step 3 - repeat iteration

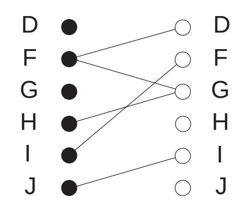


Step 1 - original vertices with no arcs

E

Step 2 - delete all arcs incident on E and redraw as shown

Step 3 - repeat iteration

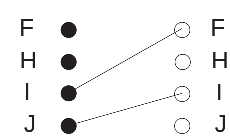


Step 1 - original vertices with no arcs

D, G

Step 2 - delete all arcs incident on D, G and redraw as shown

Step 3 - repeat iteration

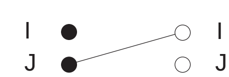


Step 1 - original vertices with no arcs

F, H

Step 2 - delete all arcs incident on F, H and redraw as shown

Step 3 - repeat iteration



Step 1 - original vertices with no arcs

I

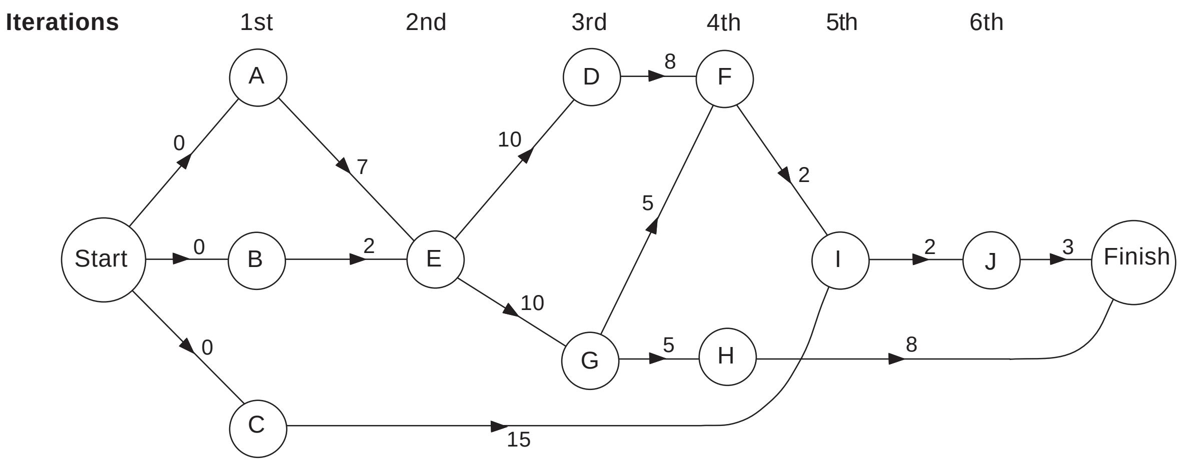
Step 2 - delete all arcs incident on I and redraw as shown

Step 3 - stop as all vertices have been chosen

So the vertices have been chosen in the following order:

A					
		D	F		
B	E			I	J
		G	H		
C					

The activity diagram as shown belowcan now be drawn.



From the 'start' vertex, draw arcs to A, B and C, the first iteration vertices, putting zero on each arc. In the original bipartite graph the shadow vertex A was joined to the original vertex E - so join A to E. Similarly join B to E and C to I.

Indicate the duration of the activity on any arc coming **from** the vertex representing the activity.

Continue in this way and complete the activity network with a 'finish' vertex into which any free vertices lead, again indicating the duration of the activity on the arc.

Note that the duration of the activity is shown on every arc **coming** from the vertex representing the activity. (So, for example, arc ED and arc EG are both given 10.)

Exercise 12A

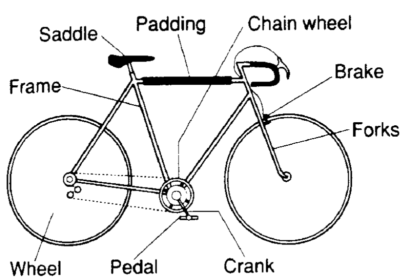
1. Use the algorithm to find the activity network for the problem in Activity 1.

2. Suppose you want to redecorate a room and put in new self-assembly units. These are the jobs that need to be done, together with the time each takes:

activity	time (in hrs)	preceded by
paint woodwork (A)	8	-
assemble units (B)	4	-
fit carpet (C)	5	hang wallpaper paint woodwork
hang wallpaper (D)	12	paint woodwork
hang curtains (E)	2	hang wallpaper paint woodwork

Complete an activity network for this problem.

3. The Spodleigh Bicycle Company is getting its assembly section ready for putting together as many bicycles as possible for the Christmas market. This diagram shows the basic components of a bicycle.



Putting together a bicycle is split up into small jobs which can be done by different people. These are:

	activity	time (mins)
A	preparation of the frame	9
B	mounting and aligning the front wheel	7
C	mounting and aligning the back wheel	7
D	attaching the chain wheel to the crank	2
E	attaching the chain wheel and crank to the frame	2
F	mounting the right pedal	8
G	mounting the left pedal	8
H	final attachments such as saddle, chain, stickers, etc.	21

The following chart shows the order of doing the jobs.

B must be after A
C must be after A
D must be after A
E must be after D
F must be after D and E
G must be after D and E
H must be after A, B, C, D, E, F and G

Draw an activity network to show this information.

4. An extension is to be built to a sports hall. Details of the activities are given below.

	activity	time (in days)
A	lay foundations	7
B	build walls	10
C	lay drains and floor	15
D	install fittings	8
E	make and fit door frames	2
F	erect roof	5
G	plaster ceiling	2
H	fit and paint doors and windows	8
I	fit gutters and pipes	2
J	paint outside	3

Some of these activities cannot be started until others have been completed:

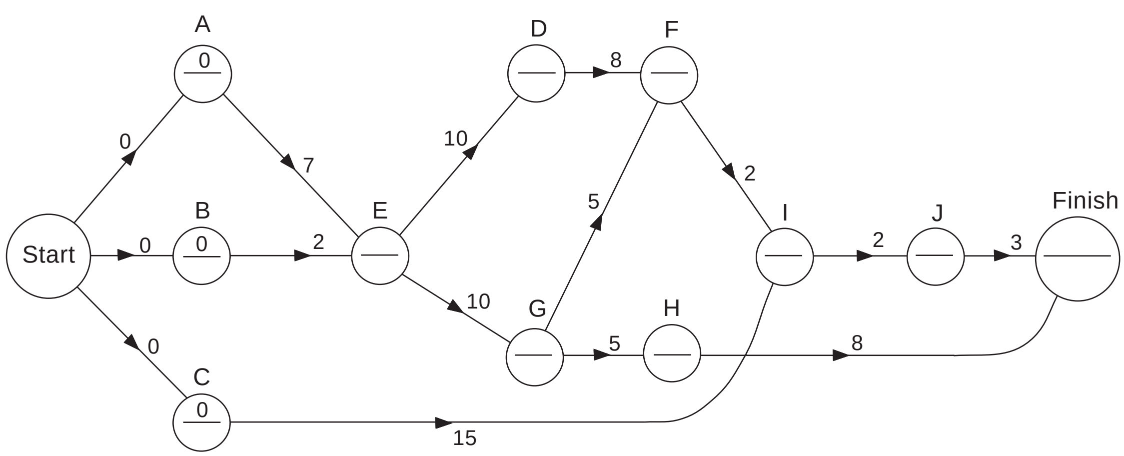
B must be after C
C must be after A
D must be after B
E must be after C
F must be after D and E
G must be after F
H must be after G
I must be after F
J must be after H

Complete an activity network for this problem.

12.3 Critical path

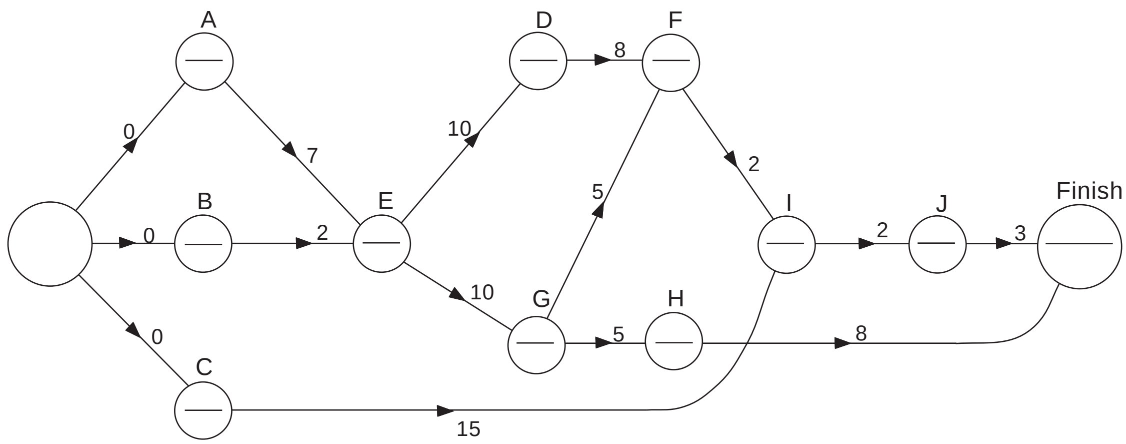
You have seen how to construct an activity network. In this section you will see how this can be used to find the **critical path**. This will first involve finding the **earliest** possible start for each activity, by going **forwards** through the network. Secondly, the **latest** possible start time for each activity is found by going **backwards** through the network. Activities which have **equal** earliest and latest start time are on the **critical path**. The technique will be illustrated using the 'garage construction' problem from Sections 12.1 and 12.2.

The activity network for this problem is shown below, where sufficient space is made at each activity node to insert two numbers.



The numbers in the top half of each circle will indicate the earliest possible starting time. So, for activities A, B and C, the number zero is inserted.

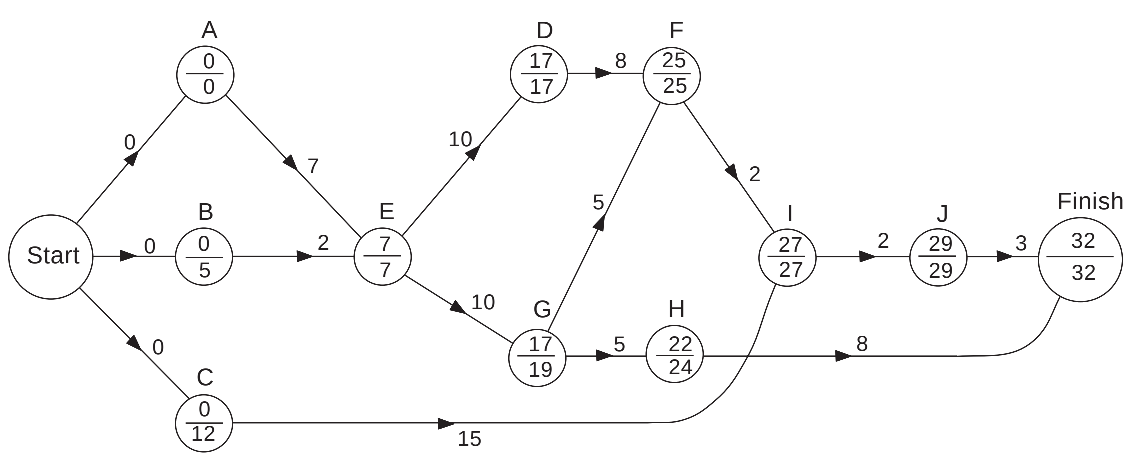
Moving forward through the network, the activity E is reached next. Since both A and B have to be completed before E can be started, the earliest start time for E is 7. This is put into the top half of the circle at E. The earliest times at D and G are then both 17, and for H, 22. Since F cannot be started until both D and G are completed, its earliest start time is 25, and consequently, 27 for I. The earliest start time for J is then 29, which gives an earliest completion time of 32.



Since 32 is the earliest possible completion time, it is also assumed to be the completion time in order to find the latest possible start times. So 32 is also put in the lower half of the 'finish' circle. Now working backwards through the network, the latest start times for each activity are as follows:

J	$32-3=29$
I	$29-2=27$
F	$27-2=25$
H	$32-8=24$
D	$25-8=17$
G	the minimum of $25-5=20$ and $24-5=19$
E	the minimum of $17-10=7$ and $19-10=9$
A	$7-7=0$
B	$7-2=5$
C	$27-15=12$

This gives a completed network as shown below.



The vertices with equal earliest and latest starting times define the **critical path**. This is clearly seen to be

A E D F I J.

Another way of identifying the critical path is to define the

float time = latest start time − earliest start time.

The information for the activities can now be summarised in the table below.

activity	start times earliest	latest	float	
A	0	0	0	←
B	0	5	5	
C	0	12	12	
E	7	7	0	←
D	17	17	0	←
G	17	19	2	
F	25	25	0	←
H	22	24	2	
I	27	27	0	←
J	29	29	0	←

So now you know that if there are enough workers the job can be completed in 32 days. The activities on the critical path (i.e. those with zero float time) must be started punctually; for example, A must start immediately, E after 7 days, F after 25 days, etc. For activities with a non-zero float time there is scope for varying their start times; for example activity G can be started any time after 17, 18 or 19 days' work. Assuming that all the work is completed on time, you will see that this does indeed give a working schedule for the construction of the garage in the minimum time of 32 days. However it does mean, for example, that on the 18th day activities D and C will definitely be in progress and G may be as well. The solution could well be affected if there was a limit to the number of workers available, but you will consider that sort of problem in the next chapter.

Is a critical path always uniquely defined?

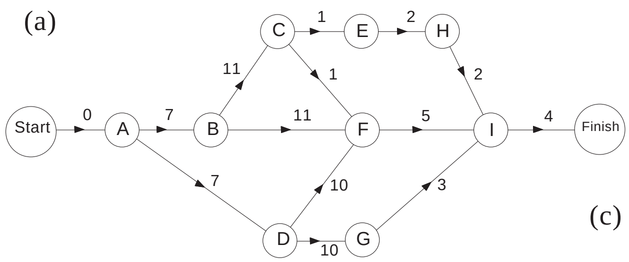
Activity 2 Bicycle construction

From the activity network for Question 3 in Exercise 12A find the critical path and the possible start times for all the activities in order to complete the job in the shortest possible time.

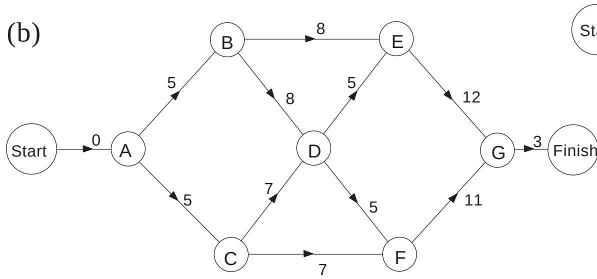
Exercise 12B

1. Find the critical paths for each of the activity networks shown below.

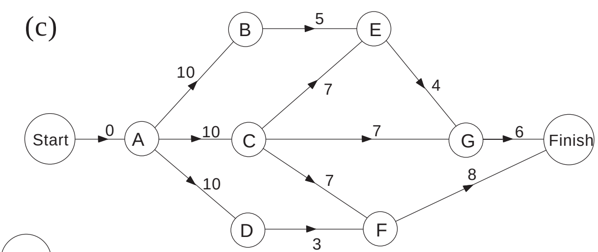
(a)



(b)



(c)



2. Find the critical path for the activity network in Question 4, Exercise 12A.

3. Your local school decides to put on a musical. These are the many jobs to be done by the organising committee, and the times they take:

A	make the costumes	6 weeks
B	rehearsals	12 weeks
C	get posters and tickets printed	3 weeks
D	get programmes printed	3 weeks
E	make scenery and props	7 weeks
F	get rights to perform the musical	2 weeks
G	choose cast	1 week
H	hire hall	1 week
I	arrange refreshments	1 week
J	organise make-up	1 week
K	decide on musical	1 week
L	organise lighting	1 week
M	dress rehearsals	2 days
N	invite local radio/press	1 day
P	choose stage hands	1 day
Q	choose programme sellers	1 day
R	choose performance dates	$\frac{1}{2}$ day
S	arrange seating	$\frac{1}{2}$ day
T	sell tickets	last 4 weeks
V	display posters	last 3 weeks

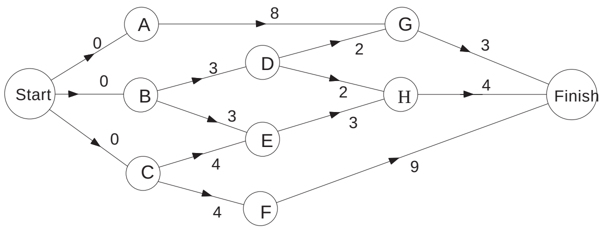
(a) Decide on the precedence relationships.

(b) Construct the activity network.

(c) Find the critical path and minimum completion time.

12.4 Miscellaneous Exercises

1. Consider the following activity network, in which the vertices represent activities and the numbers next to the arcs represent time in days.



(a) Assuming that an unlimited number of workers is available, write down:

(i) the minimum completion time of the project;

(ii)the corresponding critical path.

(b) Find the float time of activity E.

2. A project consists of ten activities, A-J. The duration (in days) of each activity, and the activities preceding each of them, are as follows:

activity	duration	preceding activities
A	10	-
B	4	-
C	8	B
D	6	C
E	8	I
F	5	-
G	10	A, D
H	2	G
I	4	-
J	10	D, F, I

Using the algorithms in Section 12.2,

(a) construct an activity network for this project;

(b) find a critical path in this activity network;

(c) find the latest starting time for each activity.

3. A project consists of eight activities whose durations are as follows:

activity	A	B	C	D	E	F	G	H
duration	4	4	3	5	4	1	6	5

The precedence relations are as follows:

B must follow A

D must follow A and C

F must follow C and E

G must follow C and E

H must follow B and D.

(a) Draw an activity network in which the activities are represented by vertices.

(b) Find a critical path by inspection, and write down the earliest and latest starting times for each activity.

4. The eleven activities A to K which make up a project are subject to the following precedence relations.

preceding activities	activity	duration
C, F, J	A	7
E	B	6
-	C	9
B, H	D	7
C, J	E	3
-	F	8
A, I	G	4
J	H	9
E, F	I	9
-	J	7
B, H, I	K	5

(a) Construct an activity network for the project.

(b) Find:

(i) the earliest starting time of each activity in the network;

(ii) the latest starting time of each activity.

(c) Calculate the float of each activity, and hence determine the critical path.

5. The activities needed to replace a broken window pane are given below.

	activity	duration (in mins)	preceding activities
A	order glass	10	-
B	collect glass	30	A
C	remove broken pane	15	B, D
D	buy putty	20	-
E	put putty in frame	3	C
F	put in new pane	2	E
G	putty outside and smooth	10	F
H	sweep up broken glass	5	C
I	clean up	5	all

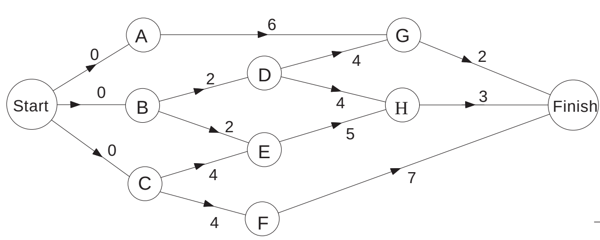
(a) Construct an activity network.

(b) What is the minimum time to complete the replacement?

(c) What is the critical path?

6. Write the major activities, duration time and precedence relationship for a real life project with which you are involved. Use the methods in this chapter to find the critical path for your project.

7. Consider the following activity network, in which the vertices represent activities and the the numbers next to the arcs represent time in weeks:



(a) Write down the minimum completion time of the project, if an unlimited number of workers is available, and the corresponding critical path.

(b) Find the float times of activities D and B.

8. A firm of landscape gardeners is asked to quote for constructing a garden on a new site. The activities involved are shown in the table.

	activity	duration (in days)	preceding activities
A	prepare site	2	-
B	build retaining wall for patio	3	A
C	lay patio * (see below)	4	A
D	lay lawn	1	A
E	lay paths	3	A B
F	erect pergola, trellis, etc.	1	A B D G
G	prepare flower beds and border	1	A B D
H	plant out	3	A B D G
I	clean up	1	all

* Note also that the patio cannot begin to be laid until 2 days after the start of the building of the retaining wall.

(a) Construct an activity network for this problem.

(b) Find the earliest and latest start time for each activity, state the minimum time for completion of the work and identify the critical path.

(c) Which activities have the greatest float time?

(AEB)

9. At 4.30 pm one day the BBC news team hear of a Government Minister resigning. They wish to prepare an item on the event for that evening's 6 o'clock news. The table below lists the jobs needed to prepare this news item, the time each job takes and the constraints on when the work can commence.

	Job	Time needed (in minutes)	Constraints
A	Interview the resigning Minister	15	Starts at 4.30 pm
B	Film Downing St.	20	None
C	Get reaction from regions	25	Cannot start until A and B are completed
D	Review possible replacements	40	Cannot start until B is completed
E	Review the Minister's career	25	Cannot start until A is completed
F	Prepare film for archives	20	Cannot start until C and E are completed
G	Edit	20	Cannot start until A, B, C, D, E and F are completed

(a) Construct an activity network for this problem and, by finding the critical path in your network, show that the news item can be ready before 6.00 pm that day.

(b) If each of the jobs A, B, C, D, E and F needs a reporter, and once a reporter has started a job that same reporter alone must complete it; explain how three reporters can have the news item ready before 6.00 pm, but that two reporters cannot. (AEB)