The Quotient Rule

A special rule, **the quotient rule**, exists for differentiating quotients of two functions. This unit illustrates this rule.

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that they become second nature.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- state the quotient rule
- differentiate quotients of functions

Contents

1. Introduction

Functions often come as **quotients**, by which we mean one function divided by another function. For example,

$$y = \frac{\cos x}{x^2}$$

We write this as $y = \dfrac{u}{v}$ where we identify u as $\cos x$ and v as x^2.

There is a formula we can use to differentiate a quotient - it is called **the quotient rule**. In this unit we will state and use the quotient rule.

2. The quotient rule

The rule states:

Key Point

The quotient rule: if $y = \dfrac{u}{v}$ then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{v\frac{\mathrm{d}u}{\mathrm{d}x} - u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$$

Let's see how the formula works when we try to differentiate $y = \dfrac{\cos x}{x^2}$.

Example

Suppose we want to differentiate $y = \dfrac{\cos x}{x^2}$.

We have identified u as $\cos x$ and v as x^2. So

$$u = \cos x \qquad v = x^2$$

We now write down the derivatives of these two functions.

$$\frac{\mathrm{d}u}{\mathrm{d}x} = -\sin x \qquad \frac{\mathrm{d}v}{\mathrm{d}x} = 2x$$

We now put all these results into the given formula:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{v\frac{\mathrm{d}u}{\mathrm{d}x} - u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$$

Quote the formula everytime so that you get to know it.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2 \cdot (-\sin x) - \cos x \cdot 2x}{(x^2)^2}$$

Notice that there is a minus sign and an x in both terms of the numerator (the top line). So we can take out a common factor of $-x$.

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{-x(x\sin x + 2\cos x)}{x^4} \\ &= \frac{-(x\sin x + 2\cos x)}{x^3}\end{aligned}$$

by cancelling the factor of x in the numerator and the denominator. We have found the required derivative.

Example

Suppose we want to differentiate $y = \dfrac{x^2+6}{2x-7}$.

We recognise this as a quotient and identify u as $x^2 + 6$ and v as $2x - 7$.

$$u = x^2 + 6 \qquad v = 2x - 7$$

Differentiating

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 2x \qquad \frac{\mathrm{d}v}{\mathrm{d}x} = 2$$

Quoting the formula:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{v\frac{\mathrm{d}u}{\mathrm{d}x} - u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$$

So

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{(2x-7)\cdot 2x - (x^2+6)\cdot 2}{(2x-7)^2} \\ &= \frac{2(2x^2 - 7x - x^2 - 6)}{(2x-7)^2} \\ &= \frac{2(x^2 - 7x - 6)}{(2x-7)^2}\end{aligned}$$

In the following Example we will use the quotient rule to establish another result.

Example

Suppose we want to differentiate $y = \tan x$.

Recall that $\tan x = \dfrac{\sin x}{\cos x}$ so we have a quotient in which

$$u = \sin x \qquad v = \cos x$$

So

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \cos x \qquad \frac{\mathrm{d}v}{\mathrm{d}x} = -\sin x$$

Quoting the formula:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{v\frac{\mathrm{d}u}{\mathrm{d}x} - u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$$

So

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\cos x \cdot \cos x - \sin x \cdot (-\sin x)}{\cos^2 x} \\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}\end{aligned}$$

The top line can be simplified using the standard result that $\cos^2 x + \sin^2 x = 1$. So

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\cos^2 x}$$

This can be written as $\sec^2 x$ because the function $\sec x$ is defined to be $\dfrac{1}{\cos x}$.

Example

Suppose we want to differentiate $y = \sec x$.

The function $\sec x$ is defined to be $\dfrac{1}{\cos x}$, that is, a quotient.
Taking

$$\begin{aligned} u &= 1 \qquad & v &= \cos x \\ \frac{\mathrm{d}u}{\mathrm{d}x} &= 0 \qquad & \frac{\mathrm{d}v}{\mathrm{d}x} &= -\sin x\end{aligned}$$

Quoting the formula:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{v\frac{\mathrm{d}u}{\mathrm{d}x} - u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$$

So

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\cos x \cdot 0 - 1 \cdot (-\sin x)}{\cos^2 x} \\ &= \frac{\sin x}{\cos^2 x}\end{aligned}$$

We can write this answer in an alternative form:

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x} \\ &= \sec x \tan x\end{aligned}$$

We now have another standard result: if $y = \sec x$ then $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \sec x \tan x$.

Key Point

if $y = \tan x$, then $\dfrac{dy}{dx} = \sec^2 x$ if $y = \sec x$, then $\dfrac{dy}{dx} = \sec x \tan x$

Exercises

Find the derivative of each of the following:

a) $\dfrac{\sin x}{x}$ b) $\dfrac{\cos x}{x^2}$ c) $\dfrac{2x+1}{3x-4}$ d) $\dfrac{3x-4}{2x+1}$

e) $\dfrac{e^{2x}}{x}$ f) $\dfrac{e^{-3x}}{x^2+1}$ g) $\dfrac{x^2-3}{2x+1}$ h) $\dfrac{2x+1}{x^2-3}$

Answers

a) $\dfrac{x\cos x - \sin x}{x^2}$ b) $\dfrac{-(x\sin x + 2\cos x)}{x^3}$ c) $\dfrac{-11}{(3x-4)^2}$ d) $\dfrac{11}{(2x+1)^2}$

e) $\dfrac{(2x-1)e^{2x}}{x^2}$ f) $\dfrac{-(3x^2+2x+3)e^{-3x}}{(x^2+1)^2}$ g) $\dfrac{2(x^2+x+3)}{(2x+1)^2}$ h) $\dfrac{-2(x^2+x+3)}{(x^2-3)^2}$