3.5 Complex numbers Summary

Polynomials

Polynomials

Polynomials are functions which just use the basic operators addition, subtraction, multiplication, division and powers like x^3.
We may refer to this as $P(x)$, for example

$$P(x) = x^2 + 3x - 2$$

Polynomials are important as we can calculate them using these basic operations.
e.g when $x = 2$,

$$\begin{aligned} x^2 + 3x - 2 &= 2^2 + 3(2) - 2 \\ &= 8 \end{aligned}$$

We can approximate other functions using polynomials, and express many functions as infinite polynomials.This is helpful both calculate them, and to understand what is meant when we use complex numbers e.g. if we know a polynomial for e^x then can calculate $e^{\sqrt{-1}}$.

Factor Theorem

One way of solving a polynomial equation is to put all the terms on one side and then factorise.
e.g.

$$\begin{aligned} x^2 + 3x &= 4 \\ x^2 + 3x - 4 &= 0 \\ (x+4)(x-1) &= 0 \end{aligned}$$

So we could have either equal to 0

$$(x+4) = 0 \quad \text{or} \quad (x-1) = 0$$
$$x = -4 \quad \text{or} \quad x = 1$$

This is similar to the **factor theorem** which states that if $(x - q)$ is a factor of a polynomial equation $P(x) = ax^3 + bx^2 + cx + d$
if $x - q = 0$ then $x = q$ and $P(q) = 0$
So if $(x - 1)$ is a factor of $P(x) = x^2 + 3x + k$ then we can work out k as $P(1) = 0$

$$\begin{aligned} 1^2 + 3(1) + k &= 0 \\ k &= -4 \end{aligned}$$

Dividing polynomials

We can divide one polynomial by another using long division.

$$\begin{array}{r l} & x + 4 \quad \text{remainder } -1 \\ x - 1 & \big)\, x^2 + 3x - 5 \\ & x^2 - x \\ & \, 4x - 5 \\ & \, \underline{4x - 4} \\ & \, -1 \end{array}$$

Note that

$$\begin{aligned} -5 - -4 &= -5 + 4 \\ &= -1 \end{aligned}$$

From this we can see that

$$\frac{x^2 + 3x - 5}{x - 1} = x + 4 + \left(\frac{-1}{x-1}\right)$$

or $\quad x^2 + 3x - 5 = (x-1)(x+4) - 1$

Remainder Theorem

We can find the remainder directly by the **remainder theorem** without long division.

If $x^2 + 3x - 5 = (x-1)P(x) + r$

Then by putting $x = 1$ we find the remainder

$$\begin{aligned} (1)^2 + 3(1) - 5 &= r \\ -1 &= r \end{aligned}$$

One way to solve a cubic equation is to guess and check solutions until no remainder. Then we can use long division to divide by the factor that we have found, which leaves a quadratic.

Surds

Surds

Surds are expressions with roots such as

$$1 + \sqrt{3} \text{ or } \sqrt{x+2}$$

usally these will be irrational as do not have an rational solutions. Note that $\sqrt{3}$ *means* $+\sqrt{3}$

Multiplying

We can expand surds like a quadratic

$$\begin{aligned} (1+\sqrt{3})(2-\sqrt{3}) &= 2 - \sqrt{3}x + 2\sqrt{3} + \sqrt{9} \\ &= 2 + \sqrt{3} + 3 \\ &= 5 + \sqrt{3} \end{aligned}$$

Simplifying

1) If **only square** in denominator, multiply top and bottom by that

$$\frac{1+\sqrt{3}}{\sqrt{3}} = \frac{(1+\sqrt{3})}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{3+\sqrt{3}}{3}$$

2) If **integer and square** in denominator, multiply by the conjugate surd to simplify as
$(a+\sqrt{b})(a-\sqrt{b}) = a^2 - b$ e.g.

$$\frac{1+\sqrt{3}}{2+\sqrt{3}} = \frac{(1+\sqrt{3})}{(2+\sqrt{3})} \times \frac{(2-\sqrt{3})}{(2-\sqrt{3})} = \frac{5+\sqrt{3}}{1}$$

Solving one type of equation

To solve an equation like

$$x + 1 = 3 + \sqrt{x+4}$$

Put surd on one side and rest on other side

$$x - 2 = \sqrt{x+4}$$

Square to remove surd

$$\begin{aligned} x^2 - 4x + 4 &= x + 4 \\ x^2 - 5x &= 0 \\ x(x-5) &= 0 \end{aligned}$$
$$x = 0 \quad \text{or} \quad x = 5$$

Solve and check if works in original equation.

Cubic formula

not in NCEA syllabus

The **cubic formula** was discovered in the 16th century by both Niccolo Fontana and Scipio del Ferro. Exploring it leads to an application for and the development of complex numbers.
For cubics it is helpful to know the expansions

$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

We can then start by simplifing a cubic
e.g. for $z^3 + 3z^2 - z = 83$ let $z = y - 1$
so $y^3 - 4y = 80$ let $y = 2x$ so $x^3 - x = 10$

To solve $x^3 - x = 10$ we can start by seeing what the solution might look like.

1) $(1+\sqrt{2})^3 = 7 + 5\sqrt{2}$
so $1 + \sqrt{2}$ is the solution to $x^3 - 5x = 2$

2) $\left(\sqrt[3]{2} + \left(\sqrt[3]{2}\right)^2\right)^3 = 6 + 6\left(\sqrt[3]{2} + \left(\sqrt[3]{2}\right)^2\right)$
so $\sqrt[3]{2} + \left(\sqrt[3]{2}\right)^2$ is the solution to $x^3 - 6x = 6$

3) For $x^3 - x = 10$ we can try a solution of the form $\sqrt[3]{b} + a\left(\sqrt[3]{b}\right)^2$ and solve for a and b etc

Plane numbers

Complex numbers

Complex or **imaginary** numbers involve $\sqrt{-1}$ which is often denoted by i.

For $3+2i$, we say 3 is the **real part** and $2i$ is the **imaginary part**.

Two plane numbers are equal if both the real and imaginary parts are equal.

We can **add** them and group like terms

$$(2\ +\ 5i)+(3\ +\ i)=(2+3)+(5-3)i$$

We can **multiply** them like a like a quadratic, using $i^2\ =\ -1$

$$(2\ +\ 5i)(3\ +\ i)=6+2i+15i+5i^2$$
$$=\ 1-5+17i$$
$$=\ -4+17i$$

Simplifying fractions

This is similar to working with surds.

1) If **only imaginary part** in denominator, multiply top and bottom by i

$$\frac{1+3i}{3i}=\frac{(1+3i)}{3i}\times\frac{i}{i}\ etc$$

2) If **real and imaginary parts** in denominator, multiply by the conjugate as

$$(a+bi)(a-bi)=a^2+b^2$$

$$\frac{1+3i}{2+31}=\frac{(1+3i)}{(2+3i)}\times\frac{(2-3i)}{(2-3i)}$$

The **conjugate** of plane number z is $\bar{z}$.

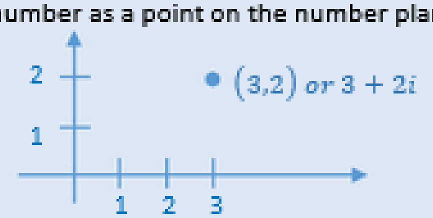
For equations like $z\bar{z}=4$ we could write in form $z=(a+bi)$ and $\bar{z}=(a-bi)$ and expand.

Plane numbers

Plane numbers is a helpful translation of 'complex' or 'imaginary' numbers.

Just as real numbers can be represented on a number line, plane numbers can be represented on a number plane.

An **Argand** diagram represents a complex number as a point on the number plane.



The $'i'$ is used to identify the second coordinate.

This is the **rectagular** form.

Loci

The **modulus** is the distance from the origin, for the plane number $z=a+bi$,

the **modulus** is $|z|=\ \sqrt{a^2+b^2}$ (or $|z|^2=z\bar{z}$)

We can graph a **circle** on the number plane, but may refer to the graph as a $\boldsymbol{locus}$ of points.

1) $|z|\ =\ 2$ describes a circle of radius 2 centred about the origin.

2) We can centre the circle about point d by

$$|z-d|\ =\ 2$$

3) $|z-3|=|z-3i|$ is the equation of line, with points the same distance from 3 and $3i$.

4) We can find the equation of the graph by letting $z\ =x+iy$

E.g. for $|z|=\ 2$, we find $x^2+y^2=4$

Polar form

Polar form

Polar form may originate from ship navigation, when the Pole Star was used to get a bearing.

Here we use an angle and distance from the origin to identify a point.

The distance is call the **modulus**. The angle is called the **argument** (arg), usually in radians.

$$Rectangular\ form\ =\ 1+i$$

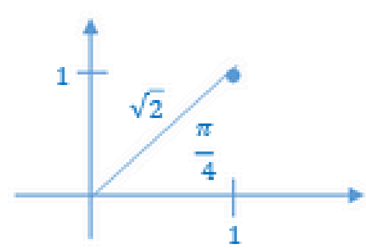
$$Polar\ form=(\sqrt{2},\frac{\pi}{4})\ or$$
$$=\sqrt{2}(\cos\frac{\pi}{4}+\ i\sin\frac{\pi}{4})$$
$$=\sqrt{2}\text{cis}\left(\frac{\pi}{4}\right)$$

There are number of ways of writing this, and main thing that a length and angle.

Converting to polar form

To convert from rectangular to polar form,

1) draw a diagram
2) calculate r using Pythagoras' theorem
3) calculate the angle using trigonometry (helpful to know the two standard triangles)



$$1+i=\ \sqrt{2}\text{cis}\left(\frac{\pi}{4}\right)$$

Multiplying and dividing

It is a lot simpler to **multiply** plane numbers together in polar form, as we only have to multiply the moduli (lengths), and add the arguments (angles). e.g.

$$3\text{cis}\left(\frac{\pi}{3}\right)\times2\text{cis}\left(\frac{\pi}{3}\right)=6\text{cis}\left(\frac{2\pi}{3}\right)$$

To **divide** two plane numbers, we divde the modulus (lengths) and subtract the arguments (angles). E.g.

$$6\text{cis}\left(\frac{2\pi}{3}\right)\div2\text{cis}\left(\frac{\pi}{3}\right)=3\text{cis}\left(\frac{\pi}{3}\right)$$

De Moivre's theorem

For raising a plane number to a power

$$(r\,\text{cis}\,\theta)^n=r^n\,\text{cis}\,n\theta$$

A solution of the nth root of a plane number is

$$\sqrt[n]{r\,\text{cis}\,\theta}=\sqrt[n]{r}\,\text{cis}\,\frac{\theta}{n}$$

Similar to square roots having two solutions, there will be n solutions.

$$\sqrt[n]{r}\,\text{cis}\left(\frac{\theta}{n}+\frac{1}{n}.2\pi\right),\ \sqrt[n]{r}\,\text{cis}\left(\frac{\theta}{n}+\frac{2}{n}.2\pi\right)\ etc$$

because when raise these to the power of n

$r\,\text{cis}(\theta+2\pi)=r\,\text{cis}(\theta)\ etc$ as $\theta+2\pi=\theta$