Higher

Mathematics

Differentiation

Contents

HSN21100
Higher Still Notes.

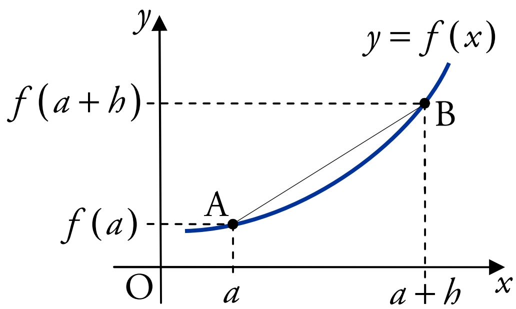
OUTCOME 3

Differentiation

1 Introduction to Differentiation

From our work on Straight Lines, we saw that the gradient (or “steepness”) of a line is constant. However, the “steepness” of other curves may not be the same at all points.

In order to measure the “steepness” of other curves, we can use lines which give an increasingly good approximation to the curve at a particular point.

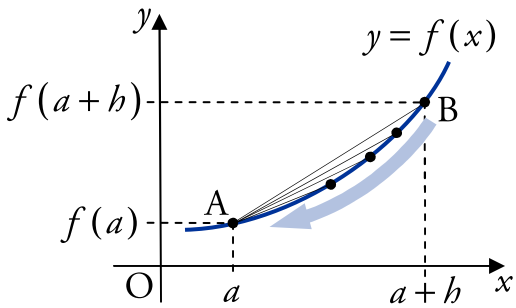


On the curve with equation $y = f(x)$, suppose point A has coordinates $(a, f(a))$.

At the point B where $x = a + h$, we have $y = f(a+h)$.

Thus the chord AB has gradient

$$\begin{aligned} m_{\text{AB}} &= \frac{f(a+h) - f(a)}{a+h-a} \\ &= \frac{f(a+h) - f(a)}{h}. \end{aligned}$$



If we let h get smaller and smaller, i.e. $h \to 0$, then B moves closer to A. This means that m_{AB} gives a better estimate of the “steepness” of the curve at the point A.

We use the notation $f'(a)$ for the “steepness” of the curve when $x = a$. So

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Given a curve with equation $y = f(x)$, an expression for $f'(x)$ is called the **derivative** and the process of finding this is called **differentiation**.

It is possible to use this definition directly to find derivates, but you will not be expected to do this. Instead, we will learn rules which allow us to quickly find derivatives for certain curves.

2 Finding the Derivative

The basic rule for differentiating $f(x)=x^n$, $n \in \mathbb{R}$, with respect to x is:

If $f(x)=x^n$ then $f'(x)=nx^{n-1}$.

Stated simply: the power (n) multiplies to the front of the x term, and the power lowers by one (giving $n-1$).

EXAMPLES

1. Given $f(x)=x^4$, find $f'(x)$.

$$f'(x)=4x^3.$$

2. Differentiate $f(x)=x^{-3}$, $x \neq 0$, with respect to x.

$$f'(x)=-3x^{-4}.$$

For an expression of the form $y=\ldots$, we denote the derivative with respect to x by $\frac{dy}{dx}$.

EXAMPLE

3. Differentiate $y=x^{-\frac{1}{3}}$, $x \neq 0$, with respect to x.

$$\frac{dy}{dx}=-\tfrac{1}{3}x^{-\frac{4}{3}}.$$

When finding the derivative of an expression with respect to x, we use the notation $\frac{d}{dx}$.

EXAMPLE

4. Find the derivative of $x^{\frac{3}{2}}$, $x \geq 0$, with respect to x.

$$\frac{d}{dx}\left(x^{\frac{3}{2}}\right)=\tfrac{3}{2}x^{\frac{1}{2}}.$$

Preparing to differentiate

It is important that before you differentiate, all brackets are multiplied out and there are no fractions with an x term in the denominator (bottom line).

For example:

$$\frac{1}{x^3}=x^{-3} \qquad \frac{3}{x^2}=3x^{-2} \qquad \frac{1}{\sqrt{x}}=x^{-\frac{1}{2}} \qquad \frac{1}{4x^5}=\tfrac{1}{4}x^{-5} \qquad \frac{5}{4\sqrt[3]{x}^2}=\tfrac{5}{4}x^{-\frac{2}{3}}.$$

EXAMPLES

1. Differentiate $\sqrt{x}$ with respect to x, where $x > 0$.

$$\sqrt{x} = x^{\frac{1}{2}}$$
$$\frac{d}{dx}\left(x^{\frac{1}{2}}\right) = \frac{1}{2}x^{-\frac{1}{2}}$$
$$= \frac{1}{2\sqrt{x}}.$$

Note
It is good practice to tidy up your answer.

2. Given $y = \frac{1}{x^2}$, where $x \neq 0$, find $\frac{dy}{dx}$.

$$y = x^{-2}$$
$$\frac{dy}{dx} = -2x^{-3}$$
$$= -\frac{2}{x^3}.$$

Terms with a coefficient

For any constant a,

$$\text{if } f(x) = a \times g(x) \quad \text{then} \quad f'(x) = a \times g'(x).$$

Stated simply: constant coefficients are carried through when differentiating.

So if $f(x) = ax^n$ then $f'(x) = anx^{n-1}$.

EXAMPLES

1. A function f is defined by $f(x) = 2x^3$. Find $f'(x)$.

$$f'(x) = 6x^2.$$

2. Differentiate $y = 4x^{-2}$ with respect to x, where $x \neq 0$.

$$\frac{dy}{dx} = -8x^{-3}$$
$$= -\frac{8}{x^3}.$$

3. Differentiate $\frac{2}{x^3}$, $x \neq 0$, with respect to x.

$$\frac{d}{dx}\left(2x^{-3}\right) = -6x^{-4}$$
$$= -\frac{6}{x^4}.$$

4. Given $y = \dfrac{3}{2\sqrt{x}}$, $x > 0$, find $\dfrac{dy}{dx}$.

$$\begin{aligned} y &= \tfrac{3}{2}x^{-\frac{1}{2}} \\ \frac{dy}{dx} &= -\tfrac{3}{4}x^{-\frac{3}{2}} \\ &= -\frac{3}{4\sqrt{x}^{3}}. \end{aligned}$$

Differentiating more than one term

The following rule allows us to differentiate expressions with several terms.

If $f(x) = g(x) + h(x)$ then $f'(x) = g'(x) + h'(x)$.

Stated simply: differentiate each term separately.

EXAMPLES

1. A function f is defined for $x \in \mathbb{R}$ by $f(x) = 3x^3 - 2x^2 + 5x$.
   Find $f'(x)$.

$$f'(x) = 9x^2 - 4x + 5.$$

2. Differentiate $y = 2x^4 - 4x^3 + 3x^2 + 6x + 2$ with respect to x.

$$\frac{dy}{dx} = 8x^3 - 12x^2 + 6x + 6.$$

Note

The derivative of an x term (e.g. $3x$, $\tfrac{1}{2}x$, $-\tfrac{3}{10}x$) is always a constant.
For example:

$$\tfrac{d}{dx}(6x) = 6, \qquad \tfrac{d}{dx}\left(-\tfrac{1}{2}x\right) = -\tfrac{1}{2}.$$

The derivative of a constant (e.g. 3, 20, π) is always zero.
For example:

$$\tfrac{d}{dx}(3) = 0, \qquad \tfrac{d}{dx}\left(-\tfrac{1}{3}\right) = 0.$$

Differentiating more complex expressions

We will now consider more complex examples where we will have to use several of the rules we have met.

EXAMPLES

1. Differentiate $y = \frac{1}{3x\sqrt{x}}$, $x > 0$, with respect to *x*.

$$y = \frac{1}{3x^{\frac{3}{2}}} = \tfrac{1}{3}x^{-\frac{3}{2}}$$

$$\frac{dy}{dx} = \tfrac{1}{3} \times -\tfrac{3}{2}x^{-\frac{5}{2}}$$

$$= -\tfrac{1}{2}x^{-\frac{5}{2}}$$

$$= -\frac{1}{2\sqrt{x}^5}.$$

Note
You need to be confident working with indices and fractions.

2. Find $\frac{dy}{dx}$ when $y = (x-3)(x+2)$.

$$y = (x-3)(x+2)$$

$$= x^2 + 2x - 3x - 6$$

$$= x^2 - x - 6$$

$$\frac{dy}{dx} = 2x - 1.$$

Remember
Before differentiating, the brackets must be multiplied out.

3. A function *f* is defined for $x \neq 0$ by $f(x) = \frac{x}{5} + \frac{1}{x^2}$. Find $f'(x)$.

$$f(x) = \tfrac{1}{5}x + x^{-2}$$

$$f'(x) = \tfrac{1}{5} - 2x^{-3}$$

$$= \tfrac{1}{5} - \frac{2}{x^3}.$$

4. Differentiate $\frac{x^4 - 3x^2}{5x}$ with respect to *x*, where $x \neq 0$.

$$\frac{x^4 - 3x^2}{5x} = \frac{x^4}{5x} - \frac{3x^2}{5x}$$

$$= \tfrac{1}{5}x^3 - \tfrac{3}{5}x$$

$$\frac{d}{dx}\left(\tfrac{1}{5}x^3 - \tfrac{3}{5}x\right) = \tfrac{3}{5}x^2 - \tfrac{3}{5}.$$

5. Differentiate $\dfrac{x^3+3x^2-6x}{\sqrt{x}}$, $x>0$, with respect to x.

$$\begin{aligned}\frac{x^3+3x^2-6x}{\sqrt{x}} &= \frac{x^3}{x^{\frac{1}{2}}}+\frac{3x^2}{x^{\frac{1}{2}}}-\frac{6x}{x^{\frac{1}{2}}}\\ &= x^{3-\frac{1}{2}}+3x^{2-\frac{1}{2}}-6x^{1-\frac{1}{2}}\\ &= x^{\frac{5}{2}}+3x^{\frac{3}{2}}-6x^{\frac{1}{2}}\end{aligned}$$

Remember
$\dfrac{x^a}{x^b}=x^{a-b}$.

$$\begin{aligned}\frac{d}{dx}\left(x^{\frac{5}{2}}+3x^{\frac{3}{2}}-6x^{\frac{1}{2}}\right) &= \tfrac{5}{2}x^{\frac{3}{2}}-\tfrac{9}{2}x^{\frac{1}{2}}-3x^{-\frac{1}{2}}\\ &= \tfrac{5}{2}\sqrt{x}^3-\tfrac{9}{2}\sqrt{x}-\frac{3}{\sqrt{x}}.\end{aligned}$$

6. Find the derivative of $y=\sqrt{x}\left(x^2+\sqrt[3]{x}\right)$, $x>0$, with respect to x.

$$y=x^{\frac{1}{2}}\left(x^2+x^{\frac{1}{3}}\right)=x^{\frac{5}{2}}+x^{\frac{5}{6}}$$

Remember
$x^a x^b=x^{a+b}$.

$$\begin{aligned}\frac{dy}{dx} &= \tfrac{5}{2}x^{\frac{3}{2}}+\tfrac{5}{6}x^{-\frac{1}{6}}\\ &= \tfrac{5}{2}\sqrt{x}^3+\frac{5}{6\sqrt[6]{x}}.\end{aligned}$$

3 Differentiating with Respect to Other Variables

So far we have differentiated functions and expressions with respect to x. However, the rules we have been using still apply if we differentiate with respect to any other variable. When modelling real-life problems we often use appropriate variable names, such as t for time and V for volume.

EXAMPLES

1. Differentiate $3t^2-2t$ with respect to t.

$$\tfrac{d}{dt}\left(3t^2-2t\right)=6t-2.$$

2. Given $A(r)=\pi r^2$, find $A'(r)$.

$$A(r)=\pi r^2$$
$$A'(r)=2\pi r.$$

Remember
π is just a constant.

When differentiating with respect to a certain variable, all other letters are treated as constants.

EXAMPLE

3. Differentiate px^2 with respect to p.

$$\tfrac{d}{dp}\left(px^2\right)=x^2.$$

Note
Since we are differentiating with respect to p, we treat x^2 as a constant.

4 Rates of Change

The derivative of a function describes its "rate of change". This can be evaluated for specific values by substituting them into the derivative.

EXAMPLES

1. Given $f(x) = 2x^5$, find the rate of change of f when $x = 3$.

$$f'(x) = 10x^4$$
$$f'(3) = 10(3)^4 = 10 \times 81 = 810.$$

2. Given $y = \frac{1}{x^{\frac{2}{3}}}$ for $x \neq 0$, calculate the rate of change of y when $x = 8$.

$$y = x^{-\frac{2}{3}}$$
$$\frac{dy}{dx} = -\frac{2}{3}x^{-\frac{5}{3}}$$
$$= -\frac{2}{3x^{\frac{5}{3}}}$$
$$= -\frac{2}{3\sqrt[3]{x}^5}.$$

At $x = 8$,
$$\frac{dy}{dx} = -\frac{2}{3\sqrt[3]{8}^5}$$
$$= -\frac{2}{3 \times 2^5}$$
$$= -\frac{2}{96}$$
$$= -\frac{1}{48}.$$

Displacement, velocity and acceleration

The velocity v of an object is defined as the rate of change of displacement s with respect to time t. That is:

$$v = \frac{ds}{dt}.$$

Also, acceleration a is defined as the rate of change of velocity with respect to time:

$$a = \frac{dv}{dt}.$$

EXAMPLE

3. A ball is thrown so that its displacement s after t seconds is given by $s(t) = 23t - 5t^2$.

Find its velocity after 2 seconds.

$$v(t) = s'(t)$$
$$= 23 - 10t$$ by differentiating $s(t) = 23t - 5t^2$ with respect to t.

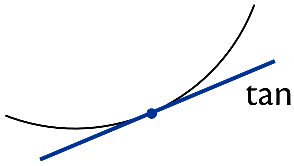
Substitute $t = 2$ into $v(t)$:

$$v(2) = 23 - 10(2) = 3.$$

After 2 seconds, the ball has velocity 3 metres per second.

5 Equations of Tangents

As we already know, the gradient of a straight line is constant. We can determine the gradient of a curve, at a particular point, by considering a straight line which touches the curve at the point. This line is called a **tangent**.



The gradient of the tangent to a curve $y = f(x)$ at $x = a$ is given by $f'(a)$.

This is the same as finding the rate of change of f at a.

To work out the equation of a tangent we use $y - b = m(x - a)$. Therefore we need to know two things about the tangent:

- a point, of which at least one coordinate will be given;
- the gradient, which is calculated by differentiating and substituting in the value of x at the required point.

EXAMPLES

1. Find the equation of the tangent to the curve with equation $y = x^2 - 3$ at the point $(2, 1)$.

We know the tangent passes through $(2, 1)$.

To find its equation, we need the gradient at the point where $x = 2$:

$$y = x^2 - 3$$

$$\frac{dy}{dx} = 2x$$

At $x = 2$, $m = 2 \times 2 = 4$.

Now we have the point $(2, 1)$ and the gradient $m = 4$, so we can find the equation of the tangent:

$$y - b = m(x - a)$$

$$y - 1 = 4(x - 2)$$

$$y - 1 = 4x - 8$$

$$4x - y - 7 = 0.$$

2. Find the equation of the tangent to the curve with equation $y = x^3 - 2x$ at the point where $x = -1$.

We need a point on the tangent. Using the given *x*-coordinate, we can find the *y*-coordinate of the point on the curve:

$$\begin{aligned} y &= x^3 - 2x \\ &= (-1)^3 - 2(-1) \\ &= -1 + 2 \\ &= 1 \qquad \text{So the point is } (-1, -1). \end{aligned}$$

We also need the gradient at the point where $x = -1$:

$$y = x^3 - 2x$$

$$\frac{dy}{dx} = 3x^2 - 2$$

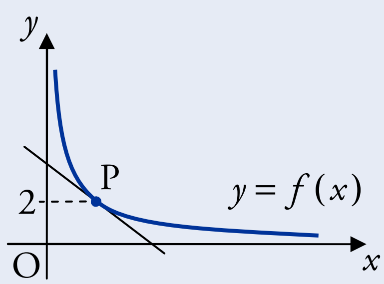
At $x = -1$, $m = 3(-1)^2 - 2 = 1$.

Now we have the point $(-1, 1)$ and the gradient $m = 1$, so the equation of the tangent is:

$$\begin{aligned} y - b &= m(x - a) \\ y - 1 &= 1(x + 1) \\ x - y + 2 &= 0. \end{aligned}$$

3. A function *f* is defined for $x > 0$ by $f(x) = \frac{1}{x}$.

Find the equation of the tangent to the curve $y = f(x)$ at P.



We need a point on the tangent. Using the given *y*-coordinate, we can find the *x*-coordinate of the point P:

$$\begin{aligned} f(x) &= 2 \\ \frac{1}{x} &= 2 \\ x &= \tfrac{1}{2} \qquad \text{So the point is } \left(\tfrac{1}{2}, 2\right). \end{aligned}$$

We also need the gradient at the point where $x = \frac{1}{2}$:

$$
\begin{aligned}
f(x) &= x^{-1} \\
f'(x) &= -x^{-2} \\
&= -\frac{1}{x^2}.
\end{aligned}
$$

At $x = \frac{1}{2}$, $m = -\dfrac{1}{\frac{1}{4}} = -4$.

Now we have the point $\left(\frac{1}{2}, 2\right)$ and the gradient $m = -4$, so the equation of the tangent is:

$$
\begin{aligned}
y - b &= m(x - a) \\
y - 2 &= -4\left(x - \tfrac{1}{2}\right) \\
y - 2 &= -4x + 2 \\
4x + y - 4 &= 0.
\end{aligned}
$$

4. Find the equation of the tangent to the curve $y = \sqrt[3]{x}^2$ at the point where $x = -8$.

We need a point on the tangent. Using the given x-coordinate, we can work out the y-coordinate:

$$
\begin{aligned}
y &= \sqrt[3]{-8}^2 \\
&= (-2)^2 \\
&= 4 \qquad \text{So the point is } (-8, 4).
\end{aligned}
$$

We also need the gradient at the point where $x = -8$:

$$
\begin{aligned}
y &= \sqrt[3]{x}^2 = x^{\frac{2}{3}} \\
\frac{dy}{dx} &= \tfrac{2}{3}x^{-\frac{1}{3}} \\
&= \frac{2}{3\sqrt[3]{x}}
\end{aligned}
$$

At $x = -8$,
$$
\begin{aligned}
m &= \frac{2}{3\sqrt[3]{8}} \\
&= \frac{2}{3 \times 2} \\
&= \tfrac{1}{3}.
\end{aligned}
$$

Now we have the point $(-8, 4)$ and the gradient $m = \frac{1}{3}$, so the equation of the tangent is:

$$
\begin{aligned}
y - b &= m(x - a) \\
y - 4 &= \tfrac{1}{3}(x + 8) \\
3y - 12 &= x + 8 \\
x - 3y + 20 &= 0.
\end{aligned}
$$

5. A curve has equation $y = \frac{1}{3}x^3 - \frac{1}{2}x^2 + 2x + 5$.
Find the coordinates of the points on the curve where the tangent has gradient 4.

The derivative gives the gradient of the tangent:

$$\frac{dy}{dx} = x^2 - x + 2.$$

We want to find where this is equal to 4:

$$\begin{aligned} x^2 - x + 2 &= 4 \\ x^2 - x - 2 &= 0 \\ (x+1)(x-2) &= 0 \\ x = -1 \ \text{ or } \ x &= 2. \end{aligned}$$

Remember
Before solving a quadratic equation you need to rearrange to get "quadratic $= 0$".

Now we can find the y-coordinates by using the equation of the curve:

$$\begin{aligned} y &= \tfrac{1}{3}(-1)^3 - \tfrac{1}{2}(-1)^2 + 2(-1) + 5 \\ &= -\tfrac{1}{3} - \tfrac{1}{2} - 2 + 5 \\ &= 3 - \tfrac{5}{6} \\ &= \tfrac{13}{6} \end{aligned}$$

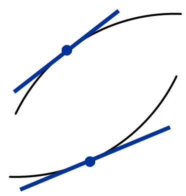
$$\begin{aligned} y &= \tfrac{1}{3}(2)^3 - \tfrac{1}{2}(2)^2 + 2(2) + 5 \\ &= \tfrac{8}{3} - \tfrac{4}{2} + 4 + 5 \\ &= 7 + \tfrac{8}{3} \\ &= \tfrac{29}{3}. \end{aligned}$$

So the points are $\left(-1, \frac{13}{6}\right)$ and $\left(2, \frac{29}{3}\right)$.

6 Increasing and Decreasing Curves

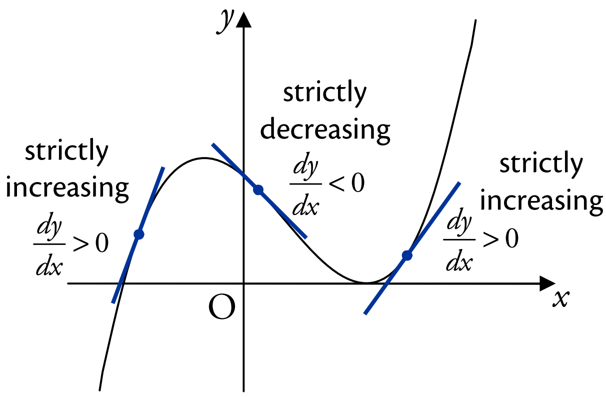
A curve is said to be **strictly increasing** when $\frac{dy}{dx} > 0$.

This is because when $\frac{dy}{dx} > 0$, tangents will slope upwards from left to right since their gradients are positive. This means the curve is also "moving upwards", i.e. strictly increasing.



Similarly:

A curve is said to be **strictly decreasing** when $\frac{dy}{dx} < 0$.



EXAMPLES

1. A curve has equation $y = 4x^2 + \frac{2}{\sqrt{x}}$.

   Determine whether the curve is increasing or decreasing at $x = 10$.

$$y = 4x^2 + 2x^{-\frac{1}{2}}$$

$$\frac{dy}{dx} = 8x - x^{-\frac{3}{2}}$$

$$= 8x - \frac{1}{\sqrt{x}^3}.$$

When $x = 10$, $\frac{dy}{dx} = 8 \times 10 - \frac{1}{\sqrt{10}^3}$

$$= 80 - \frac{1}{10\sqrt{10}}$$

$$> 0.$$

Note

$\frac{1}{10\sqrt{10}} < 1.$

Since $\frac{dy}{dx} > 0$, the curve is increasing when $x = 10$.

2. Show that the curve $y = \frac{1}{3}x^3 + x^2 + x - 4$ is never decreasing.

$$\begin{aligned}\frac{dy}{dx} &= x^2 + 2x + 1\\ &= (x+1)^2\\ &\geq 0.\end{aligned}$$

Remember
The result of squaring any number is always greater than, or equal to, zero.

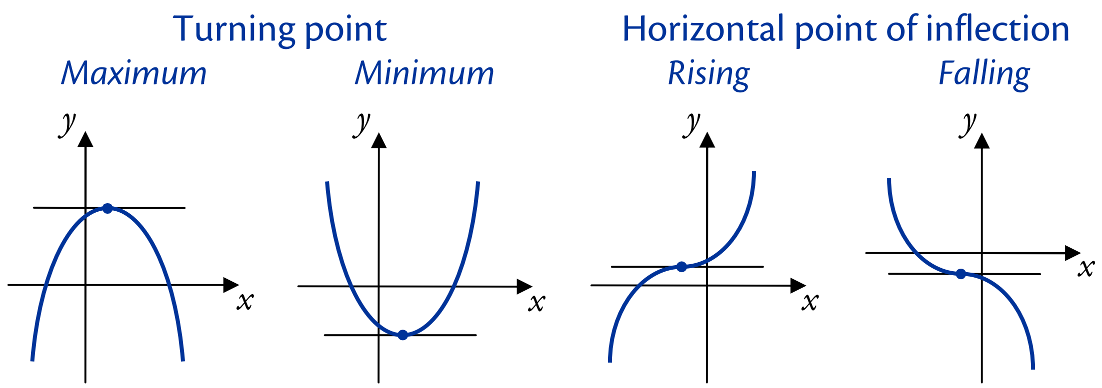
Since $\frac{dy}{dx}$ is never less than zero, the curve is never decreasing.

7 Stationary Points

At some points, a curve may be neither increasing nor decreasing – we say that the curve is **stationary** at these points.

This means that the gradient of the tangent to the curve is zero at stationary points, so we can find them by solving $f'(x) = 0$ or $\frac{dy}{dx} = 0$.

The four possible stationary points are:



A stationary point's nature (type) is determined by the behaviour of the graph to its left and right. This is often done using a "nature table".

8 Determining the Nature of Stationary Points

To illustrate the method used to find stationary points and determine their nature, we will do this for the graph of $f(x) = 2x^3 - 9x^2 + 12x + 4$.

Step 1

Differentiate the function.

$$f'(x) = 6x^2 - 18x + 12$$

Step 2

Find the stationary values by solving $f'(x) = 0$.

$$\begin{aligned} f'(x) &= 0 \\ 6x^2 - 18x + 12 &= 0 \\ 6(x^2 - 3x + 2) &= 0 \quad (\div 6) \\ (x-1)(x-2) &= 0 \\ x = 1 \ \text{ or } \ x &= 2 \end{aligned}$$

Step 3

Find the y-coordinates of the stationary points.

$f(1) = 9$ so $(1, 9)$ is a stat. pt.
$f(2) = 8$ so $(2, 8)$ is a stat. pt.

Step 4

Write the stationary values in the top row of the nature table, with arrows leading in and out of them.

x	$\to$ 1 $\to$	$\to$ 2 $\to$
$f'(x)$		
Graph		

Step 5

Calculate $f'(x)$ for the values in the table, and record the results. This gives the gradient at these x values, so zeros confirm that stationary points exist here.

x	$\to$ 1 $\to$	$\to$ 2 $\to$
$f'(x)$	0	0
Graph		

Step 6

Calculate $f'(x)$ for values slightly lower and higher than the stationary values and record the sign in the second row, e.g.: $f'(0.8) > 0$ so enter $+$ in the first cell.

x	$\to$ 1 $\to$	$\to$ 2 $\to$
$f'(x)$	$+$ 0 $-$	$-$ 0 $+$
Graph		

Step 7

We can now sketch the graph near the stationary points:
$+$ means the graph is increasing and
$-$ means the graph is decreasing.

x	$\to$ 1 $\to$	$\to$ 2 $\to$
$f'(x)$	$+$ 0 $-$	$-$ 0 $+$
Graph	/ ‾ \	\ _ /

Step 8

The nature of the stationary points can then be concluded from the sketch.

$(1, 9)$ is a max. turning point.
$(2, 8)$ is a min. turning point.

EXAMPLES

1. A curve has equation $y = x^3 - 6x^2 + 9x - 4$.

   Find the stationary points on the curve and determine their nature.

Given $y = x^3 - 6x^2 + 9x - 4$,

$$\frac{dy}{dx} = 3x^2 - 12x + 9.$$

Stationary points exist where $\frac{dy}{dx} = 0$:

$$3x^2 - 12x + 9 = 0$$
$$3\left(x^2 - 4x + 3\right) = 0 \quad (\div 3)$$
$$x^2 - 4x + 3 = 0$$
$$(x-1)(x-3) = 0$$
$$x - 1 = 0 \quad \text{or} \quad x - 3 = 0$$
$$x = 1 \qquad\qquad x = 3.$$

When $x = 1$,

$$\begin{aligned} y &= (1)^3 - 6(1)^2 + 9(1) - 4 \\ &= 1 - 6 + 9 - 4 \\ &= 0. \end{aligned}$$

Therefore the point is $(1,0)$.

When $x = 3$,

$$\begin{aligned} y &= (3)^3 - 6(3)^2 + 9(3) - 4 \\ &= 27 - 54 + 27 - 4 \\ &= -4. \end{aligned}$$

Therefore the point is $(3,-4)$.

Nature:

x	$\rightarrow$	1	$\rightarrow$	$\rightarrow$	3	$\rightarrow$
$\frac{dy}{dx}$	+	0	–	–	0	+
Graph	/	—	\	\	_	/

So $(1,0)$ is a maximum turning point,

$(3,-4)$ is a minimum turning point.

2. Find the stationary points of $y = 4x^3 - 2x^4$ and determine their nature.

Given $y = 4x^3 - 2x^4$,

$$\frac{dy}{dx} = 12x^2 - 8x^3.$$

Stationary points exist where $\frac{dy}{dx} = 0$:

$$12x^2 - 8x^3 = 0$$

$$4x^2(3 - 2x) = 0$$

$$4x^2 = 0 \quad \text{or} \quad 3 - 2x = 0$$

$$x = 0 \qquad\qquad x = \tfrac{3}{2}.$$

When $x = 0$,

$$y = 4(0)^3 - 2(0)^4 = 0.$$

Therefore the point is $(0, 0)$.

When $x = \frac{3}{2}$,

$$y = 4\left(\tfrac{3}{2}\right)^3 - 2\left(\tfrac{3}{2}\right)^4 = \tfrac{27}{2} - \tfrac{81}{8} = \tfrac{27}{8}.$$

Therefore the point is $\left(\frac{3}{2}, \frac{27}{8}\right)$.

Nature:

x	$\rightarrow$	0	$\rightarrow$	$\rightarrow$	$\frac{3}{2}$	$\rightarrow$
$\frac{dy}{dx}$	$+$	0	$+$	$+$	0	$-$
Graph	/	—	/	/	—	\

So $(0, 0)$ is a rising point of inflection,

$\left(\frac{3}{2}, \frac{27}{8}\right)$ is a maximum turning point.

3. A curve has equation $y = 2x + \dfrac{1}{x}$ for $x \neq 0$. Find the x-coordinates of the stationary points on the curve and determine their nature.

Given $y = 2x + x^{-1}$,

$$\begin{aligned}\frac{dy}{dx} &= 2 - x^{-2} \\ &= 2 - \frac{1}{x^2}.\end{aligned}$$

Stationary points exist where $\dfrac{dy}{dx} = 0$:

$$\begin{aligned}2 - \frac{1}{x^2} &= 0 \\ 2x^2 &= 1 \\ x^2 &= \tfrac{1}{2} \\ x &= \pm\tfrac{1}{\sqrt{2}}.\end{aligned}$$

Nature:

x	$\rightarrow$	$-\frac{1}{\sqrt{2}}$	$\rightarrow$	$\rightarrow$	$\frac{1}{\sqrt{2}}$	$\rightarrow$
$\frac{dy}{dx}$	$+$	0	$-$	$-$	0	$+$
Graph	/	—	\	\	_	/

So the point where $x = -\frac{1}{\sqrt{2}}$ is a maximum turning point and the point where $x = \frac{1}{\sqrt{2}}$ is a minimum turning point.

9 Curve Sketching

In order to sketch a curve, we first need to find the following:

- x-axis intercepts (roots) – solve $y = 0$;
- y-axis intercept – find y for $x = 0$;
- stationary points and their nature.

EXAMPLE

Sketch the curve with equation $y = 2x^3 - 3x^2$.

y-axis intercept, i.e. $x = 0$:

$$\begin{aligned} y &= 2(0)^3 - 3(0)^2 \\ &= 0. \end{aligned}$$

Therefore the point is $(0, 0)$.

x-axis intercepts i.e. $y = 0$:

$$2x^3 - 3x^2 = 0$$
$$x^2(2x - 3) = 0$$

$$\begin{array}{ccc} x^2 = 0 & \text{or} & 2x - 3 = 0 \\ x = 0 & & x = \frac{3}{2} \\ (0, 0) & & \left(\frac{3}{2}, 0\right). \end{array}$$

Given $y = 2x^3 - 3x^2$,

$$\frac{dy}{dx} = 6x^2 - 6x.$$

Stationary points exist where $\frac{dy}{dx} = 0$:

$$6x^2 - 6x = 0$$
$$6x(x - 1) = 0$$

$$\begin{array}{ccc} 6x = 0 & \text{or} & x - 1 = 0 \\ x = 0 & & x = 1. \end{array}$$

When $x = 0$,

$$\begin{aligned} y &= 2(0)^3 - 3(0)^2 \\ &= 0. \end{aligned}$$

Therefore the point is $(0, 0)$.

When $x = 1$,

$$\begin{aligned} y &= 2(1)^3 - 3(1)^2 \\ &= 2 - 3 \\ &= -1. \end{aligned}$$

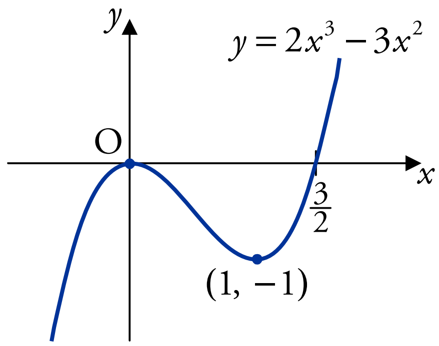
Therefore the point is $(1, -1)$.

Nature:

x	$\rightarrow$	0	$\rightarrow$	$\rightarrow$	1	$\rightarrow$
$\frac{dy}{dx}$	+	0	−	−	0	+
Graph	/	—	\	\	_	/

$(0, 0)$ is a maximum turning point.

$(1, -1)$ is a minimum turning point.

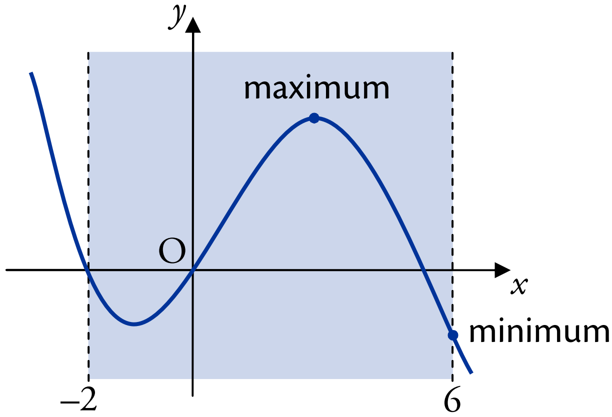


10 Closed Intervals

Sometimes it is necessary to restrict the part of the graph we are looking at using a **closed interval** (also called a restricted domain).

The maximum and minimum values of a function can either be at its stationary points *or* at the end points of a closed interval.

Below is a sketch of a curve with the closed interval $-2 \le x \le 6$ shaded.



Notice that the minimum value occurs at one of the end points in this example. It is important to check for this whenever we are dealing with a closed interval.

EXAMPLE

A function f is defined for $-1 \le x \le 4$ by $f(x) = 2x^3 - 5x^2 - 4x + 1$.

Find the maximum and minimum value of $f(x)$.

Given $f(x) = 2x^3 - 5x^2 - 4x + 1$,

$$f'(x) = 6x^2 - 10x - 4.$$

Stationary points exist where $f'(x) = 0$:

$$6x^2 - 10x - 4 = 0$$

$$2(3x^2 - 5x - 2) = 0$$

$$(x-2)(3x+1) = 0$$

$$x - 2 = 0 \quad \text{or} \quad 3x + 1 = 0$$

$$x = 2 \qquad\qquad x = -\tfrac{1}{3}.$$

To find coordinates of stationary points:

$$\begin{aligned} f(2) &= 2(2)^3 - 5(2)^2 - 4(2) + 1 \\ &= 16 - 20 - 8 + 1 \\ &= -11. \end{aligned}$$

Therefore the point is $(2, -11)$.

$$\begin{aligned} f\left(-\tfrac{1}{3}\right) &= 2\left(-\tfrac{1}{3}\right)^3 - 5\left(-\tfrac{1}{3}\right)^2 - 4\left(-\tfrac{1}{3}\right) + 1 \\ &= 2\left(-\tfrac{1}{27}\right) - 5\left(\tfrac{1}{9}\right) - 4\left(\tfrac{1}{3}\right) + 1 \\ &= -\tfrac{2}{27} - \tfrac{5}{9} + \tfrac{4}{3} + 1 \\ &= \tfrac{46}{27}. \end{aligned}$$

Therefore the point is $\left(-\tfrac{1}{3}, \tfrac{46}{27}\right)$.

Nature:

x	$\rightarrow$	$-\frac{1}{3}$	$\rightarrow$	$\rightarrow$	2	$\rightarrow$
$f'(x)$	$+$	0	$-$	$-$	0	$+$
Graph	/	—	\	\	_	/

$\left(-\tfrac{1}{3}, \tfrac{46}{27}\right)$ is a max. turning point.

$(2, -11)$ is a min. turning point.

Points at extremities of closed interval:

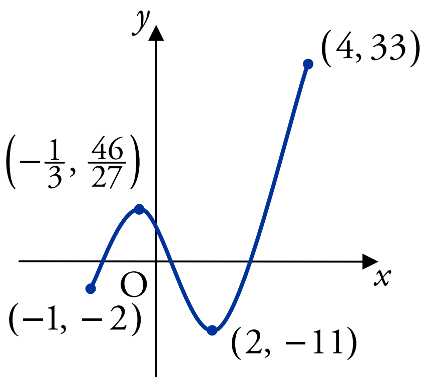
$$\begin{aligned} f(-1) &= 2(-1)^3 - 5(-1)^2 - 4(-1) + 1 \\ &= -2 - 5 + 4 + 1 \\ &= -2. \end{aligned}$$

Therefore the point is $(-1, -2)$.

$$\begin{aligned} f(4) &= 2(4)^3 - 5(4)^2 - 4(4) + 1 \\ &= 128 - 80 - 16 + 1 \\ &= 33. \end{aligned}$$

Therefore the point is $(4, 33)$.

Now we can make a sketch:



Note
A sketch may help you to decide on the correct answer, but it is not required in the exam.

The maximum value is 33 which occurs when $x = 4$.

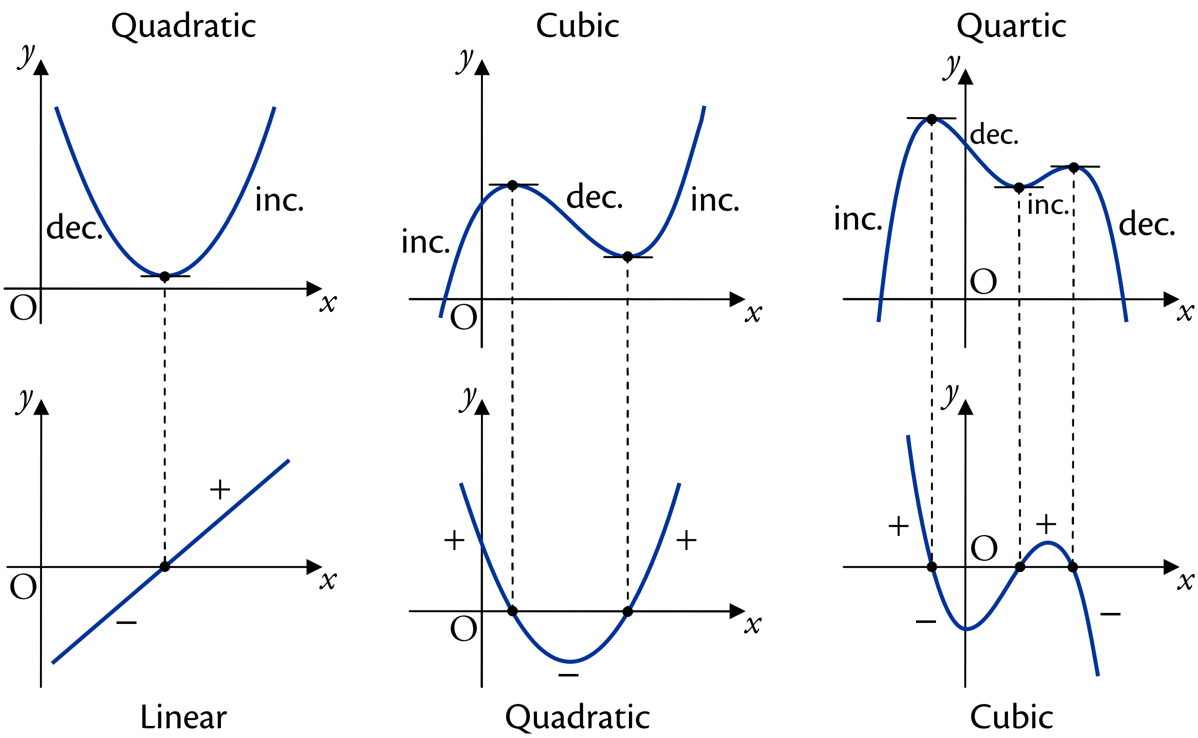
The minimum value is -11 which occurs when $x = 2$.

11 Graphs of Derivatives

The derivative of an x^n term is an x^{n-1} term – the power lowers by one. For example, the derivative of a cubic (where x^3 is the highest power of x) is a quadratic (where x^2 is the highest power of x).

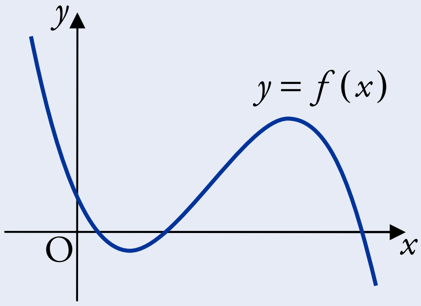
When drawing a derived graph:

- All stationary points of the original curve become roots (i.e. lie on the x-axis) on the graph of the derivative.
- Wherever the curve is strictly decreasing, the derivative is negative. So the graph of the derivative will lie below the x-axis – it will take negative values.
- Wherever the curve is strictly increasing, the derivative is positive. So the graph of the derivative will lie above the x-axis – it will take positive values.



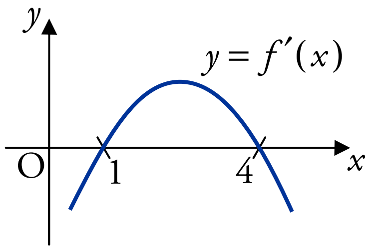
EXAMPLE

The curve $y = f(x)$ shown below is a cubic. It has stationary points where $x = 1$ and $x = 4$.



Sketch the graph of $y = f'(x)$.

Since $y = f(x)$ has stationary points at $x = 1$ and $x = 4$, the graph of $y = f'(x)$ crosses the x-axis at $x = 1$ and $x = 4$.



Note
The curve is increasing between the stationary points so the derivative is positive there.

12 Optimisation

In the section on closed intervals, we saw that it is possible to find maximum and minimum values of a function.

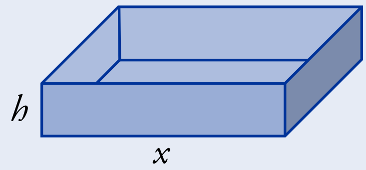
This is often useful in applications; for example a company may have a function $P(x)$ which predicts the profit if £x is spent on raw materials – the management would be very interested in finding the value of x which gave the maximum value of $P(x)$.

The process of finding these optimal values is called **optimisation**.

Sometimes you will have to find the appropriate function before you can start optimisation.

EXAMPLE

1. Small plastic trays, with open tops and square bases, are being designed. They must have a volume of 108 cubic centimetres.



The internal length of one side of the base is x centimetres, and the internal height of the tray is h centimetres.

(a) Show that the total internal surface area A of one tray is given by

$$A = x^2 + \frac{432}{x}.$$

(b) Find the dimensions of the tray using the least amount of plastic.

(a) $$\begin{aligned}\text{Volume} &= \text{area of base} \times \text{height} \\ &= x^2 h.\end{aligned}$$

We are told that the volume is 108 cm^3, so:

$$\begin{aligned}\text{Volume} &= 108 \\ x^2 h &= 108 \\ h &= \frac{108}{x^2}.\end{aligned}$$

Let A be the surface area for a particular value of x:

$$A = x^2 + 4xh.$$

We have $h = \dfrac{108}{x^2}$, so:

$$\begin{aligned}A &= x^2 + 4x\left(\frac{108}{x^2}\right) \\ &= x^2 + \frac{432}{x}.\end{aligned}$$

(b) The smallest amount of plastic is used when the surface area is minimised.

$$\frac{dA}{dx} = 2x - \frac{432}{x^2}.$$

Stationary points occur when $\dfrac{dA}{dx} = 0$:

$$\begin{aligned}2x - \frac{432}{x^2} &= 0 \\ x^3 &= 216 \\ x &= 6.\end{aligned}$$

Nature:

x	$\rightarrow$	6	$\rightarrow$
$\frac{dA}{dx}$	$-$	0	$+$
Graph	\	—	/

So the minimum surface area occurs when $x = 6$. For this value of x:

$$h = \frac{108}{6^2} = 3.$$

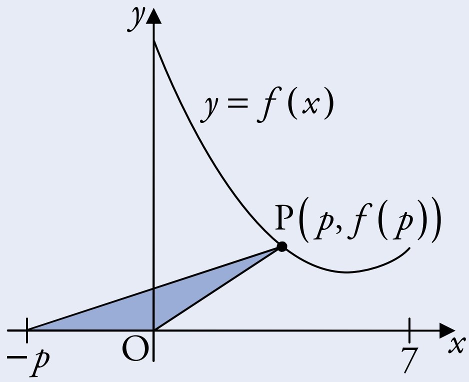
So a length and depth of 6 cm and a height of 3 cm uses the least amount of plastic.

Optimisation with closed intervals

In practical situations, there may be bounds on the values we can use. For example, the company from before might only have £100 000 available to spend on raw materials. We would need to take this into account when optimising.

Recall from the section on Closed Intervals that the maximum and minimum values of a function can occur at turning points *or* the endpoints of a closed interval.

2. The point P lies on the graph of $f(x) = x^2 - 12x + 45$, between $x = 0$ and $x = 7$.



A triangle is formed with vertices at the origin, P and $(-p, 0)$.

(a) Show that the area, A square units, of this triangle is given by

$$A = \tfrac{1}{2}p^3 - 6p^2 + \tfrac{45}{2}p.$$

(b) Find the greatest possible value of A and the corresponding value of p for which it occurs.

(a) The area of the triangle is

$$\begin{aligned} A &= \tfrac{1}{2} \times \text{base} \times \text{height} \\ &= \tfrac{1}{2} \times p \times f(p) \\ &= \tfrac{1}{2}p\left(p^2 - 12p + 45\right) \\ &= \tfrac{1}{2}p^3 - 6p^2 + \tfrac{45}{2}p. \end{aligned}$$

(b) The greatest value occurs at a stationary point or an endpoint.

At stationary points $\frac{dA}{dp} = 0$:

$$\begin{aligned}\frac{dA}{dp} = \tfrac{3}{2}p^2 - 12p + \tfrac{45}{2} &= 0 \\ 3p^2 - 24p + 45 &= 0 \\ p^2 - 8p + 15 &= 0 \\ (p-3)(p-5) &= 0 \\ p = 3 \quad \text{or} \quad p &= 5.\end{aligned}$$

Now evaluate A at the stationary points and endpoints:

- when $p = 0$, $A = 0$;
- when $p = 3$, $A = \tfrac{1}{2} \times 3^3 - 6 \times 3^2 + \tfrac{45}{2} \times 3 = 27$;
- when $p = 5$, $A = \tfrac{1}{2} \times 5^3 - 6 \times 5^2 + \tfrac{45}{2} \times 5 = 25$;
- when $p = 7$, $A = \tfrac{1}{2} \times 7^3 - 6 \times 7^2 + \tfrac{45}{2} \times 7 = 35$.

So the greatest possible value of A is 35, which occurs when $p = 7$.

Higher

Mathematics

Integration

Contents

OUTCOME 2

Integration

1 Indefinite Integrals

In **integration**, our aim is to "undo" the process of differentiation. Later we will see that integration is a useful tool for evaluating areas and solving a special type of equation.

We have already seen how to differentiate polynomials, so we will now look at how to undo this process. The basic technique is:

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + c \qquad n \neq -1,\ c \text{ is the constant of integration.}$$

Stated simply: raise the power (n) by one (giving $n+1$), divide by the new power ($n+1$), and add the constant of integration (c).

EXAMPLES

1. Find $\int x^2 \, dx$.

$$\int x^2 \, dx = \frac{x^3}{3} + c = \tfrac{1}{3}x^3 + c.$$

2. Find $\int x^{-3} \, dx$.

$$\int x^{-3} \, dx = \frac{x^{-2}}{-2} + c = -\frac{1}{2x^2} + c.$$

3. Find $\int x^{\frac{5}{4}} \, dx$.

$$\int x^{\frac{5}{4}} \, dx = \frac{x^{\frac{9}{4}}}{\frac{9}{4}} + c = \tfrac{4}{9}x^{\frac{9}{4}} + c.$$

- We use the symbol $\int$ for integration.
- The $\int$ must be used with "dx" in the examples above, to indicate that we are integrating with respect to x.
- The constant of integration is included to represent any constant term in the original expression, since this would have been zeroed by differentiation.
- Integrals with a constant of integration are called **indefinite integrals**.

Checking the answer

Since integration and differentiation are reverse processes, if we differentiate our answer we should get back to what we started with.

For example, if we differentiate our answer to Example 1 above, we do get back to the expression we started with.

$$\begin{tikzcd} \frac{1}{3}x^3 + c \arrow[rr, bend left, "\text{differentiate}"] & & x^2 \arrow[ll, bend left, "\text{integrate}"] \end{tikzcd}$$

Integrating terms with coefficients

The above technique can be extended to:

$$\int ax^n \, dx = a\int x^n \, dx = \frac{ax^{n+1}}{n+1} + c \qquad n \neq -1,\ a \text{ is a constant.}$$

Stated simply: raise the power (n) by one (giving $n+1$), divide by the new power ($n+1$), and add on c.

EXAMPLES

4. Find $\int 6x^3 \, dx$.

$$\begin{aligned} \int 6x^3 \, dx &= \frac{6x^4}{4} + c \\ &= \tfrac{3}{2}x^4 + c. \end{aligned}$$

5. Find $\int 4x^{-\frac{3}{2}} \, dx$.

$$\begin{aligned} \int 4x^{-\frac{3}{2}} \, dx &= \frac{4x^{-\frac{1}{2}}}{-\frac{1}{2}} + c \\ &= -8x^{-\frac{1}{2}} + c \\ &= -\frac{8}{\sqrt{x}} + c. \end{aligned}$$

Note

It can be easy to confuse integration and differentiation, so remember:

$$\int x \, dx = \tfrac{1}{2}x^2 + c \qquad\qquad \int k \, dx = kx + c.$$

Other variables

Just as with differentiation, we can integrate with respect to any variable.

EXAMPLES

6. Find $\int 2p^{-5}\,dp$.

$$\begin{aligned}\int 2p^{-5}\,dp &= \frac{2p^{-4}}{-4}+c\\ &= -\frac{1}{2p^{4}}+c.\end{aligned}$$

Note
dp tells us to integrate with respect to *p*.

7. Find $\int p\,dx$.

$$\begin{aligned}&\int p\,dx\\ &= px+c.\end{aligned}$$

Note
Since we are integrating with respect to *x*, we treat *p* as a constant.

Integrating several terms

The following rule is used to integrate an expression with several terms:

$$\int\left(f(x)+g(x)\right)dx = \int f(x)\,dx + \int g(x)\,dx.$$

Stated simply: integrate each term separately.

EXAMPLES

8. Find $\int\left(3x^{2}-2x^{\frac{1}{2}}\right)dx$.

$$\begin{aligned}\int\left(3x^{2}-2x^{\frac{1}{2}}\right)dx &= \frac{3x^{3}}{3}-\frac{2x^{\frac{3}{2}}}{\frac{3}{2}}+c\\ &= x^{3}-\frac{4x^{\frac{3}{2}}}{3}+c\\ &= x^{3}-\tfrac{4}{3}\sqrt{x^{3}}+c.\end{aligned}$$

9. Find $\int\left(4x^{-\frac{5}{8}}+3x+7\right)dx$.

$$\begin{aligned}\int\left(4x^{-\frac{5}{8}}+3x+7\right)dx &= \frac{4x^{\frac{3}{8}}}{\frac{3}{8}}+\frac{3x^{2}}{2}+7x+c\\ &= \tfrac{8}{3}\times 4x^{\frac{3}{8}}+\tfrac{3}{2}x^{2}+7x+c\\ &= \tfrac{32}{3}x^{\frac{3}{8}}+\tfrac{3}{2}x^{2}+7x+c.\end{aligned}$$

2 Preparing to Integrate

As with differentiation, it is important that before integrating all brackets are multiplied out and there are no fractions with an x term in the denominator (bottom line), for example:

$$\frac{1}{x^3} = x^{-3} \qquad \frac{3}{x^2} = 3x^{-2} \qquad \frac{1}{\sqrt{x}} = x^{-\frac{1}{2}} \qquad \frac{1}{4x^5} = \tfrac{1}{4}x^{-5} \qquad \frac{5}{4\sqrt[3]{x}^2} = \tfrac{5}{4}x^{-\frac{2}{3}}.$$

EXAMPLES

1. Find $\int \frac{dx}{x^2}$ for $x \neq 0$.

$\int \frac{dx}{x^2}$ is just a short way of writing $\int \frac{1}{x^2}\,dx$, so:

$$\begin{aligned}\int \frac{dx}{x^2} = \int \frac{1}{x^2}\,dx = \int x^{-2}dx &= \frac{x^{-1}}{-1} + c \\ &= -\frac{1}{x} + c.\end{aligned}$$

2. Find $\int \frac{dx}{\sqrt{x}}$ for $x > 0$.

$$\begin{aligned}\int \frac{dx}{\sqrt{x}} = \int \frac{1}{x^{\frac{1}{2}}}dx = \int x^{-\frac{1}{2}}dx &= \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c \\ &= 2\sqrt{x} + c.\end{aligned}$$

3. Find $\int \frac{7}{2p^2}\,dp$ where $p \neq 0$.

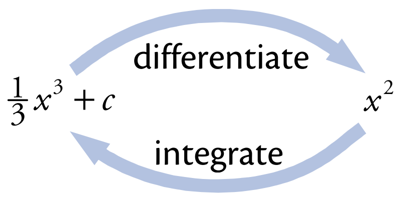
$$\begin{aligned}\int \frac{7}{2p^2}\,dp = \int \tfrac{7}{2}p^{-2}\,dp \\ &= \tfrac{7}{2} \times \frac{p^{-1}}{-1} + c \\ &= -\frac{7}{2p} + c.\end{aligned}$$

4. Find $\displaystyle\int \frac{3x^5 - 5x}{4}\, dx$.

$$
\begin{aligned}
\int \frac{3x^5 - 5x}{4}\, dx &= \int \left(\tfrac{3}{4}x^5 - \tfrac{5}{4}x \right) dx \\
&= \frac{3x^6}{4 \times 6} - \frac{5x^2}{4 \times 2} + c \\
&= \tfrac{3}{24}x^6 - \tfrac{5}{8}x^2 + c \\
&= \tfrac{1}{8}x^6 - \tfrac{5}{8}x^2 + c.
\end{aligned}
$$

3 Differential Equations

A **differential equation** is an equation involving derivatives, e.g. $\dfrac{dy}{dx} = x^2$.

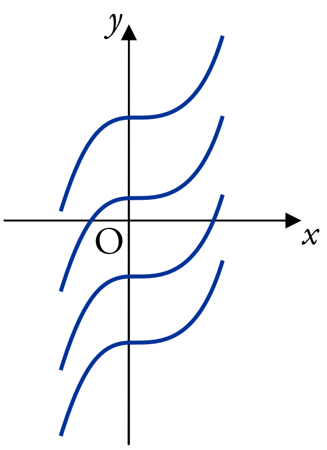


A solution of a differential equation is an expression for the original function; in this case $y = \frac{1}{3}x^3 + c$ is a solution.

In general, we obtain solutions using integration:

$$y = \int \frac{dy}{dx}\, dx \qquad \text{or} \qquad f(x) = \int f'(x)\, dx.$$

This will result in a **general solution** since we can choose the value of c, the constant of integration.



The general solution corresponds to a “family” of curves, each with a different value for c.

The graph to the left illustrates some of the curves $y = \frac{1}{3}x^3 + c$ for particular values of c.

If we have additional information about the function (such as a point its graph passes through), we can find the value of c and obtain a **particular solution**.

EXAMPLES

1. The graph of $y = f(x)$ passes through the point $(3, -4)$.

   If $\frac{dy}{dx} = x^2 - 5$, express y in terms of x.

$$\begin{aligned} y &= \int \frac{dy}{dx}\, dx \\ &= \int \left(x^2 - 5\right)\, dx \\ &= \tfrac{1}{3}x^3 - 5x + c. \end{aligned}$$

We know that when $x = 3$, $y = -4$ so we can find c:

$$\begin{aligned} y &= \tfrac{1}{3}x^3 - 5x + c \\ -4 &= \tfrac{1}{3}(3)^3 - 5(3) + c \\ -4 &= 9 - 15 + c \\ c &= 2 \end{aligned}$$

So $y = \tfrac{1}{3}x^3 - 5x + 2$.

2. The function f, defined on a suitable domain, is such that

   $f'(x) = x^2 + \dfrac{1}{x^2} + \tfrac{2}{3}$.

   Given that $f(1) = 4$, find a formula for $f(x)$ in terms of x.

$$\begin{aligned} f(x) &= \int f'(x)\, dx \\ &= \int \left(x^2 + \frac{1}{x^2} + \tfrac{2}{3}\right) dx \\ &= \int \left(x^2 + x^{-2} + \tfrac{2}{3}\right) dx \\ &= \tfrac{1}{3}x^3 - x^{-1} + \tfrac{2}{3}x + c \\ &= \tfrac{1}{3}x^3 - \frac{1}{x} + \tfrac{2}{3}x + c. \end{aligned}$$

We know that $f(1) = 4$, so we can find c:

$$\begin{aligned} f(x) &= \tfrac{1}{3}x^3 - \frac{1}{x} + \tfrac{2}{3}x + c \\ 4 &= \tfrac{1}{3}(1)^3 - \tfrac{1}{1} + \tfrac{2}{3}(1) + c \\ 4 &= \tfrac{1}{3} - 1 + \tfrac{2}{3} + c \\ c &= 4. \end{aligned}$$

So $f(x) = \tfrac{1}{3}x^3 - \dfrac{1}{x} + \tfrac{2}{3}x + 4$.

4 Definite Integrals

If $F(x)$ is an integral of $f(x)$, then we define:

$$\int_a^b f(x)\,dx = \big[F(x)\big]_a^b = F(b) - F(a)$$

where a and b are called the **limits** of the integral.

Stated simply:

- Work out the integral as normal, leaving out the constant of integration.
- Evaluate the integral for $x = b$ (the upper limit value).
- Evaluate the integral for $x = a$ (the lower limit value).
- Subtract the lower limit value from the upper limit value.

Since there is no constant of integration and we are calculating a numerical value, this is called a **definite integral**.

EXAMPLES

1. Find $\int_1^3 5x^2\,dx$.

$$\begin{aligned}\int_1^3 5x^2\,dx &= \left[\frac{5x^3}{3}\right]_1^3\\ &= \left(\frac{5(3)^3}{3}\right) - \left(\frac{5(1)^3}{3}\right)\\ &= 5\times 3^2 - \tfrac{5}{3}\\ &= 45 - \tfrac{5}{3} = 43\tfrac{1}{3}.\end{aligned}$$

2. Find $\int_0^2 \left(x^3 + 3x^2\right)dx$.

$$\begin{aligned}\int_0^2 \left(x^3 + 3x^2\right)dx &= \left[\frac{x^4}{4} + \frac{3x^3}{3}\right]_0^2\\ &= \left[\frac{x^4}{4} + x^3\right]_0^2\\ &= \left(\frac{2^4}{4} + 2^3\right) - \left(\frac{0^4}{4} + 0^3\right)\\ &= \tfrac{16}{4} + 8 - 0\\ &= 4 + 8 = 12.\end{aligned}$$

3. Find $\displaystyle\int_{-1}^{4} \frac{4}{x^3}\, dx$.

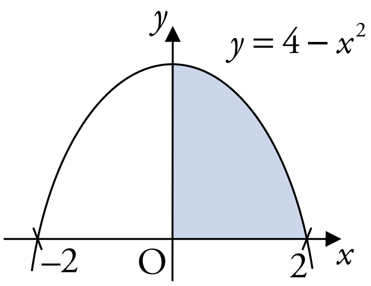
$$\begin{aligned}
\int_{-1}^{4} \frac{4}{x^3}\, dx &= \int_{-1}^{4} 4x^{-3}\, dx \\
&= \left[\frac{4x^{-2}}{-2}\right]_{-1}^{4} \\
&= \left[-\frac{2}{x^2}\right]_{-1}^{4} \\
&= \left(-\frac{2}{4^2}\right) - \left(-\frac{2}{(-1)^2}\right) \\
&= -\tfrac{2}{16} + 2 = 1\tfrac{7}{8}.
\end{aligned}$$

5 Geometric Interpretation of Integration

We will now consider the meaning of integration in the context of areas.

Consider
$$\begin{aligned}
\int_0^2 \left(4 - x^2\right) dx &= \left[4x - \tfrac{1}{3}x^3\right]_0^2 \\
&= \left(8 - \tfrac{8}{3}\right) - 0 \\
&= 5\tfrac{1}{3}.
\end{aligned}$$

On the graph of $y = 4 - x^2$:



The shaded area is given by $\int_0^2 \left(4 - x^2\right) dx$.

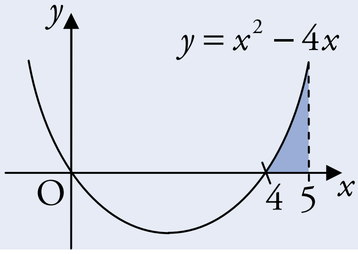
Therefore the shaded area is $5\frac{1}{3}$ square units.

In general, the area enclosed by the graph $y = f(x)$ and the x-axis, between $x = a$ and $x = b$, is given by

$$\int_a^b f(x)\, dx.$$

EXAMPLE

1. The graph of $y = x^2 - 4x$ is shown below. Calculate the shaded area.

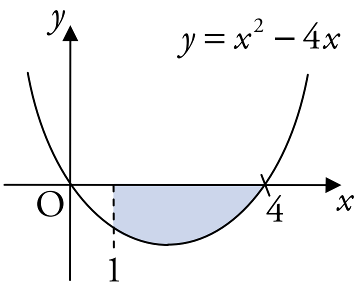


$$\int_4^5 \left(x^2 - 4x\right) dx = \left[\frac{x^3}{3} - \frac{4x^2}{2}\right]_4^5$$
$$= \left(\frac{5^3}{3} - 2(5)^2\right) - \left(\frac{4^3}{3} - 2(4)^2\right)$$
$$= \tfrac{125}{3} - 50 - \tfrac{64}{3} + 32$$
$$= \tfrac{61}{3} - 18$$
$$= 2\tfrac{1}{3}.$$

So the shaded area is $2\frac{1}{3}$ square units.

Areas below the *x*-axis

Care needs to be taken if part or all of the area lies below the x-axis. For example if we look at the graph of $y = x^2 - 4$:



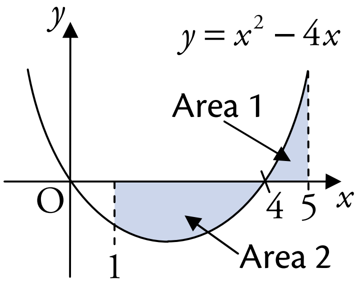
The shaded area is given by

$$\int_1^4 \left(x^2 - 4x\right) dx = \left[\frac{x^3}{3} - \frac{4x^2}{2}\right]_1^4$$
$$= \left(\frac{4^3}{3} - 2(4)^2\right) - \left(\tfrac{1}{3} - 2\right)$$
$$= \tfrac{64}{3} - 32 - \tfrac{1}{3} + 2$$
$$= \tfrac{63}{3} - 30 = 21 - 30 = -9.$$

In this case, the negative indicates that the area is below the x-axis, as can be seen from the diagram. The area is therefore 9 square units.

Areas above and below the x-axis

Consider the graph from the example above, with a different shaded area:



From the working above, the total shaded area is:

$$\text{Area 1} + \text{Area 2} = 2\tfrac{1}{3} + 9 = 11\tfrac{1}{3} \text{ square units.}$$

Using the method from above, we might try to calculate the shaded area as follows:

$$\begin{aligned}\int_1^5 \left(x^2 - 4x\right) dx &= \left[\frac{x^3}{3} - \frac{4x^2}{2}\right]_1^5 \\ &= \left(\frac{5^3}{3} - 2(5)^2\right) - \left(\tfrac{1}{3} - 2\right) \\ &= \tfrac{125}{3} - 50 - \tfrac{1}{3} + 2 \\ &= \tfrac{124}{3} - 48 = -6\tfrac{2}{3}.\end{aligned}$$

Clearly this shaded area is not $6\tfrac{2}{3}$ square units since we already found it to be $11\tfrac{1}{3}$ square units. This problem arises because Area 1 is above the x-axis, while Area 2 is below.

To find the true area, we needed to evaluate two integrals:

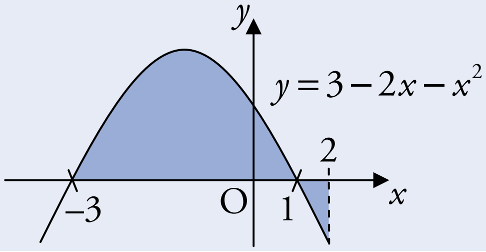
$$\int_1^4 \left(x^2 - 4x\right) dx \qquad \text{and} \qquad \int_4^5 \left(x^2 - 4x\right) dx.$$

We then found the total shaded area by adding the two areas together.

We must take care to do this whenever the area is split up in this way.

EXAMPLES

2. Calculate the shaded area shown in the diagram below.



To calculate the area from $x = -3$ to $x = 1$:

$$\begin{aligned}
\int_{-3}^{1} \left(3 - 2x - x^2\right) dx &= \left[3x - \frac{2x^2}{2} - \frac{x^3}{3}\right]_{-3}^{1} \\
&= \left[3x - x^2 - \tfrac{1}{3}x^3\right]_{-3}^{1} \\
&= \left(3(1) - (1)^2 - \tfrac{1}{3}(1)^3\right) - \left(3(-3) - (-3)^2 - \tfrac{1}{3}(-3)^3\right) \\
&= \left(3 - 1 - \tfrac{1}{3}\right) - (-9 - 9 + 9) \\
&= 3 - 1 - \tfrac{1}{3} + 9 \\
&= 10\tfrac{2}{3} \qquad \text{So the area is } 10\tfrac{2}{3} \text{ square units.}
\end{aligned}$$

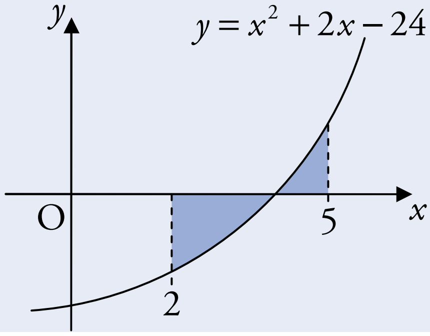
We have already carried out the integration, so we can just substitute in new limits to work out the area from $x = 1$ to $x = 2$:

$$\begin{aligned}
\int_{1}^{2} \left(3 - 2x - x^2\right) dx &= \left[3x - x^2 - \tfrac{1}{3}x^3\right]_{1}^{2} \\
&= \left(3(2) - (2)^2 - \tfrac{1}{3}(2)^3\right) - \left(3(1) - (1)^2 - \tfrac{1}{3}(1)^3\right) \\
&= \left(6 - 4 - \tfrac{8}{3}\right) - \left(3 - 1 - \tfrac{1}{3}\right) \\
&= 2 - \tfrac{8}{3} - 2 + \tfrac{1}{3} \\
&= -2\tfrac{1}{3}. \qquad \text{So the area is } 2\tfrac{1}{3} \text{ square units.}
\end{aligned}$$

Remember
The negative sign just indicates that the area lies below the axis.

So the total shaded area is $10\frac{2}{3} + 2\frac{1}{3} = 13$ square units.

3. Calculate the shaded area shown in the diagram below.



First, we need to calculate the root between $x = 2$ and $x = 5$:

$$x^2 + 2x - 24 = 0$$
$$(x-4)(x+6) = 0$$
$$x = 4 \quad \text{or} \quad x = -6.$$

So the root is $x = 4$

To calculate the area from $x = 2$ to $x = 4$:

$$\begin{aligned}
\int_2^4 (x^2 + 2x - 24)\, dx &= \left[\frac{x^3}{3} + \frac{2x^2}{2} - 24x\right]_2^4 \\
&= \left[\tfrac{1}{3}x^3 + x^2 - 24x\right]_2^4 \\
&= \left(\tfrac{1}{3}(4)^3 + (4)^2 - 24(4)\right) - \left(\tfrac{1}{3}(2)^3 + (2)^2 - 24(2)\right) \\
&= \left(\tfrac{64}{3} + 16 - 96\right) - \left(\tfrac{8}{3} + 4 - 48\right) \\
&= \tfrac{56}{3} - 36 \\
&= -17\tfrac{1}{3} \qquad \text{So the area is } 17\tfrac{1}{3} \text{ square units.}
\end{aligned}$$

To calculate the area from $x = 4$ to $x = 5$:

$$\begin{aligned}
\int_4^5 (x^2 + 2x - 24)\, dx &= \left[\tfrac{1}{3}x^3 + x^2 - 24x\right]_4^5 \\
&= \left(\tfrac{1}{3}(5)^3 + (5)^2 - 24(5)\right) - \left(\tfrac{1}{3}(4)^3 + (4)^2 - 24(4)\right) \\
&= \left(\tfrac{125}{3} + 25 - 120\right) - \left(\tfrac{64}{3} + 16 - 96\right) \\
&= \tfrac{61}{3} - 15 \\
&= 5\tfrac{1}{3} \qquad \text{So the area is } 5\tfrac{1}{3} \text{ square units.}
\end{aligned}$$

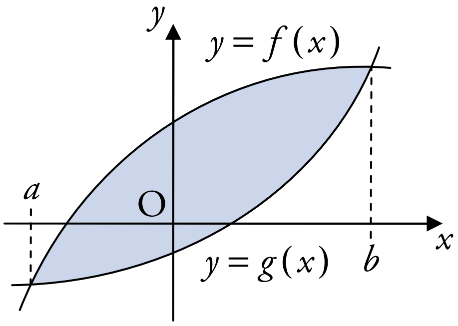
So the total shaded area is $17\tfrac{1}{3} + 5\tfrac{1}{3} = 22\tfrac{2}{3}$ square units.

6 Areas between Curves

The area between two curves between $x = a$ and $x = b$ is calculated as:

$$\int_a^b (\text{upper curve} - \text{lower curve})\, dx \text{ square units.}$$

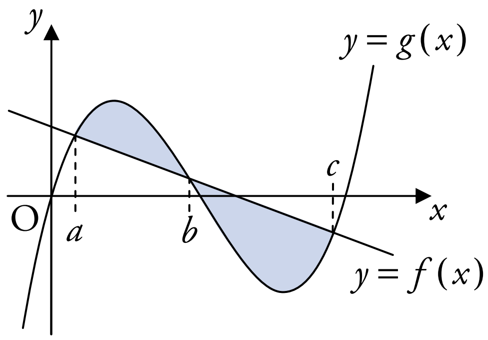
So for the shaded area shown below:



The area is $\int_a^b (f(x) - g(x))\, dx$ square units.

When dealing with areas between curves, areas above and below the x-axis do not need to be calculated separately.

However, care must be taken with more complicated curves, as these may give rise to more than one closed area. These areas must be evaluated separately. For example:



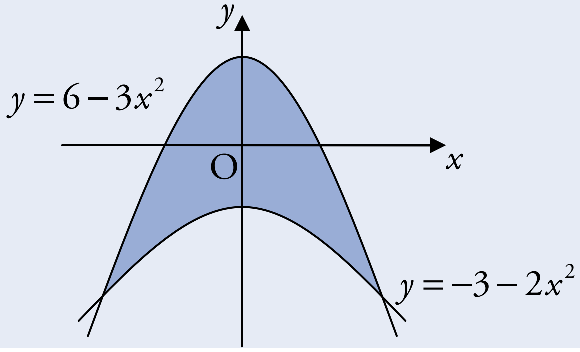
In this case we apply $\int_a^b (\text{upper curve} - \text{lower curve})\, dx$ to each area.

So the shaded area is given by:

$$\int_a^b (g(x) - f(x))\, dx + \int_b^c (f(x) - g(x))\, dx.$$

EXAMPLES

1. Calculate the shaded area enclosed by the curves with equations $y = 6 - 3x^2$ and $y = -3 - 2x^2$.



To work out the points of intersection, equate the curves:

$$\begin{aligned} 6 - 3x^2 &= -3 - 2x^2 \\ 6 + 3 - 3x^2 + 2x^2 &= 0 \\ 9 - x^2 &= 0 \\ (3 + x)(3 - x) &= 0 \\ x = -3 \quad &\text{or} \quad x = 3. \end{aligned}$$

Set up the integral and simplify:

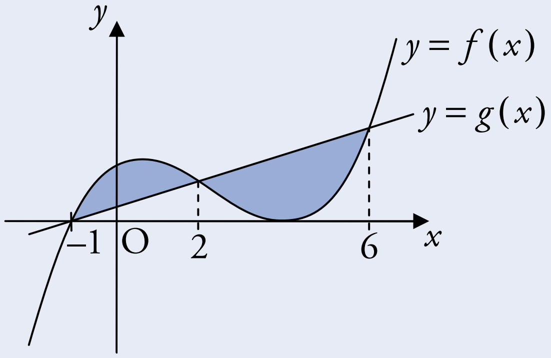
$$\begin{aligned} &\int_{-3}^{3} (\text{upper curve} - \text{lower curve})\, dx \\ &= \int_{-3}^{3} \left((6 - 3x^2) - (-3 - 2x^2) \right) dx \\ &= \int_{-3}^{3} \left(6 - 3x^2 + 3 + 2x^2 \right) dx \\ &= \int_{-3}^{3} \left(9 - x^2 \right) dx. \end{aligned}$$

Carry out integration:

$$\begin{aligned} \int_{-3}^{3} \left(9 - x^2 \right) dx &= \left[9x - \frac{x^3}{3} \right]_{-3}^{3} \\ &= \left(9(3) - \frac{(3)^3}{3} \right) - \left(9(-3) - \frac{(-3)^3}{3} \right) \\ &= \left(27 - \tfrac{27}{3} \right) - \left(-27 + \tfrac{27}{3} \right) \\ &= 27 - 9 + 27 - 9 \\ &= 36. \end{aligned}$$

Therefore the shaded area is 36 square units.

2. Two functions are defined for $x \in \mathbb{R}$ by $f(x) = x^3 - 7x^2 + 8x + 16$ and $g(x) = 4x + 4$. The graphs of $y = f(x)$ and $y = g(x)$ are shown below.



Calculate the shaded area.

Since the shaded area is in two parts, we apply $\int_a^b (\text{upper} - \text{lower})\,dx$ twice.

Area from $x = -1$ to $x = 2$:

$$
\begin{aligned}
&\int_{-1}^{2} (\text{upper} - \text{lower})\,dx \\
&= \int_{-1}^{2} \left(x^3 - 7x^2 + 8x + 16 - (4x + 4)\right) dx \\
&= \int_{-1}^{2} \left(x^3 - 7x^2 + 4x + 12\right) dx \\
&= \left[\frac{x^4}{4} - \frac{7x^3}{3} + \frac{4x^2}{2} + 12x\right]_{-1}^{2} \\
&= \left(\frac{2^4}{4} - \frac{7\times 2^3}{3} + 2\times 2^2 + 12\times 2\right) - \left(\frac{(-1)^4}{4} - \frac{7(-1)^3}{3} + 2(-1)^2 + 12(-1)\right) \\
&= \left(4 - \tfrac{56}{3} + 8 + 24\right) - \left(\tfrac{1}{4} + \tfrac{7}{3} + 2 - 12\right) \\
&= \frac{99}{4} \\
&= 24\tfrac{3}{4}.
\end{aligned}
$$

Note
The curve is at the top of this area.

So the first area is $24\frac{3}{4}$ square units.

Area from $x = 2$ to $x = 6$:

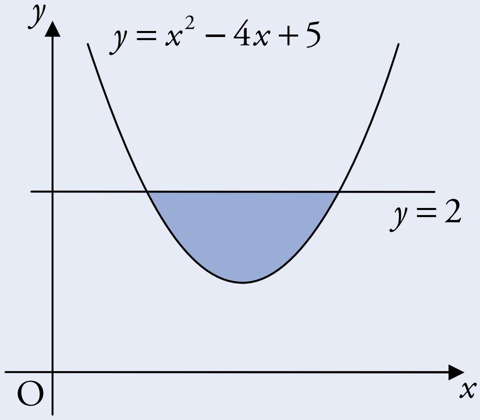
$$\int_2^6 (\text{upper} - \text{lower})\, dx$$
$$= \int_2^6 \left(4x - 4 - \left(x^3 - 7x^2 + 8x + 16\right)\right) dx$$
$$= \int_2^6 \left(-x^3 + 7x^2 - 4x - 12\right) dx$$
$$= \left[-\frac{x^4}{4} + \frac{7x^3}{3} - \frac{4x^2}{2} - 12x\right]_2^6$$
$$= \left(-\frac{6^4}{4} + \frac{7\times 6^3}{3} - \frac{4\times 6^2}{2} - 12\times 6\right) - \left(-\frac{2^4}{4} + \frac{7\times 2^3}{3} - \frac{4\times 2^2}{2} - 12\times 2\right)$$
$$= (-324 + 504 - 72 - 72) - \left(-4 + \tfrac{56}{3} - 8 - 24\right)$$
$$= \tfrac{160}{3}$$
$$= 53\tfrac{1}{3}.$$

Note
The straight line is at the top of this area.

So the second area is $53\frac{1}{3}$ square units.

So the total shaded area is $24\frac{3}{4} + 53\frac{1}{3} = 78\frac{1}{12}$ square units.

3. A trough is 2 metres long. A cross-section of the trough is shown below.



The cross-section is part of the parabola with equation $y = x^2 - 4x + 5$.

Find the volume of the trough.

To work out the points of intersection, equate the curve and the line:

$$x^2 - 4x + 5 = 2$$
$$x^2 - 4x + 3 = 0$$
$$(x-1)(x-3) = 0 \text{ so } x = 1 \text{ or } x = 3.$$

Set up the integral and integrate:

$$\begin{aligned}\int_1^3 (\text{upper} - \text{lower})\, dx &= \int_1^3 \left(2 - \left(x^2 - 4x + 5\right)\right) dx \\ &= \int_1^3 \left(-x^2 + 4x - 3\right) dx \\ &= \left[-\frac{x^3}{3} + \frac{4x^2}{2} - 3x\right]_1^3 \\ &= \left(-\frac{(3)^3}{3} + 2(3)^2 - 3(3)\right) - \left(-\frac{(1)^3}{3} + 2(1)^2 - 3(1)\right) \\ &= (-9 + 18 - 9) - \left(-\tfrac{1}{3} + 2 - 3\right) \\ &= 0 + \tfrac{1}{3} - 2 + 3 \\ &= \tfrac{4}{3} \\ &= 1\tfrac{1}{3}.\end{aligned}$$

Therefore the shaded area is $1\frac{1}{3}$ square units.

$$\begin{aligned}\text{Volume} &= \text{cross-sectional area} \times \text{length} \\ &= \tfrac{4}{3} \times 2 \\ &= \tfrac{8}{3} \quad = 2\tfrac{2}{3}.\end{aligned}$$

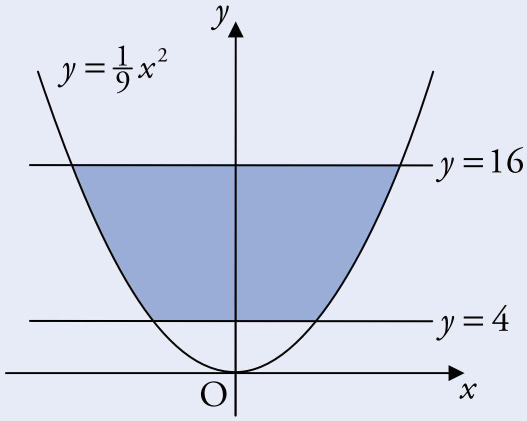
Therefore the volume of the trough is $2\frac{2}{3}$ cubic units.

7 Integrating along the *y*-axis

For some problems, it may be easier to find a shaded area by integrating with respect to y rather than x.

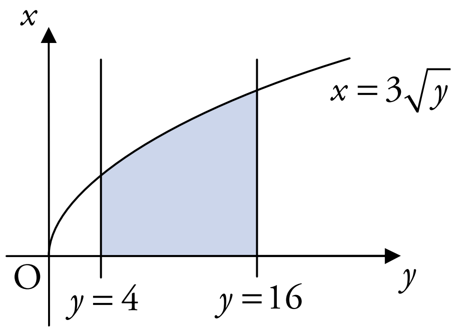
EXAMPLE

The curve with equation $y = \frac{1}{9}x^2$ is shown in the diagram below.



Calculate the shaded area which lies between $y = 4$ and $y = 16$.

We have
$$\begin{aligned} y &= \tfrac{1}{9}x^2 \\ 9y &= x^2 \\ x &= \pm\sqrt{9y} \\ x &= \pm 3\sqrt{y}. \end{aligned}$$



The shaded area in the diagram to the right is given by:

$$\begin{aligned} \int_4^{16} 3\sqrt{y}\,dy &= \int_4^{16} 3y^{\frac{1}{2}}\,dy \\ &= \left[\frac{3y^{\frac{3}{2}}}{\frac{3}{2}}\right]_4^{16} \\ &= \left[2\sqrt{y}^{\,3}\right]_4^{16} \\ &= 2\sqrt{16}^{\,3} - 2\sqrt{4}^{\,3} \\ &= 2\times 64 - 2\times 8 \\ &= 112. \end{aligned}$$

Since this is half of the required area, the total shaded area is 224 square units.